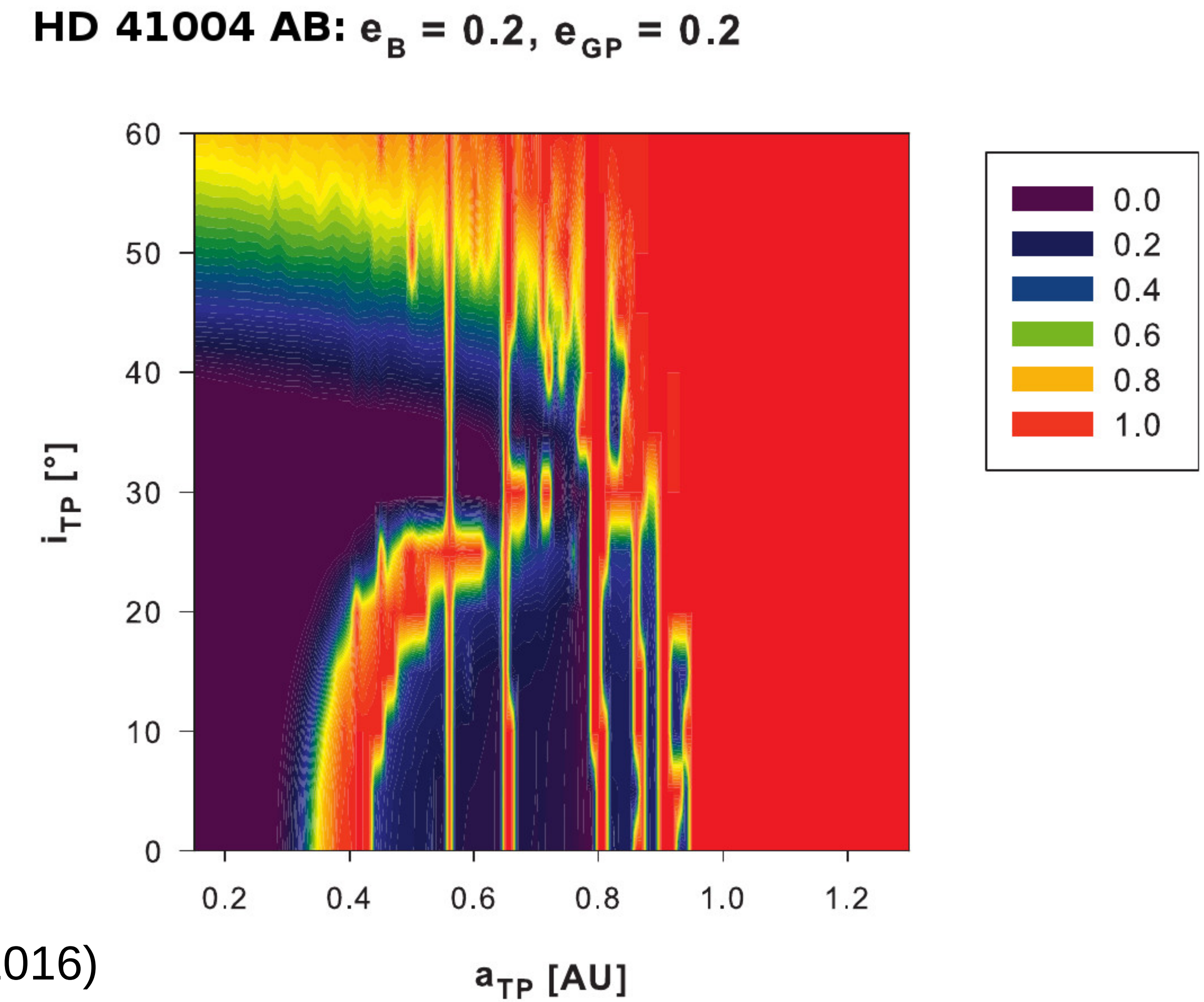
Semi-analytical Method for Secular Resonances

Part 1

Topics overview

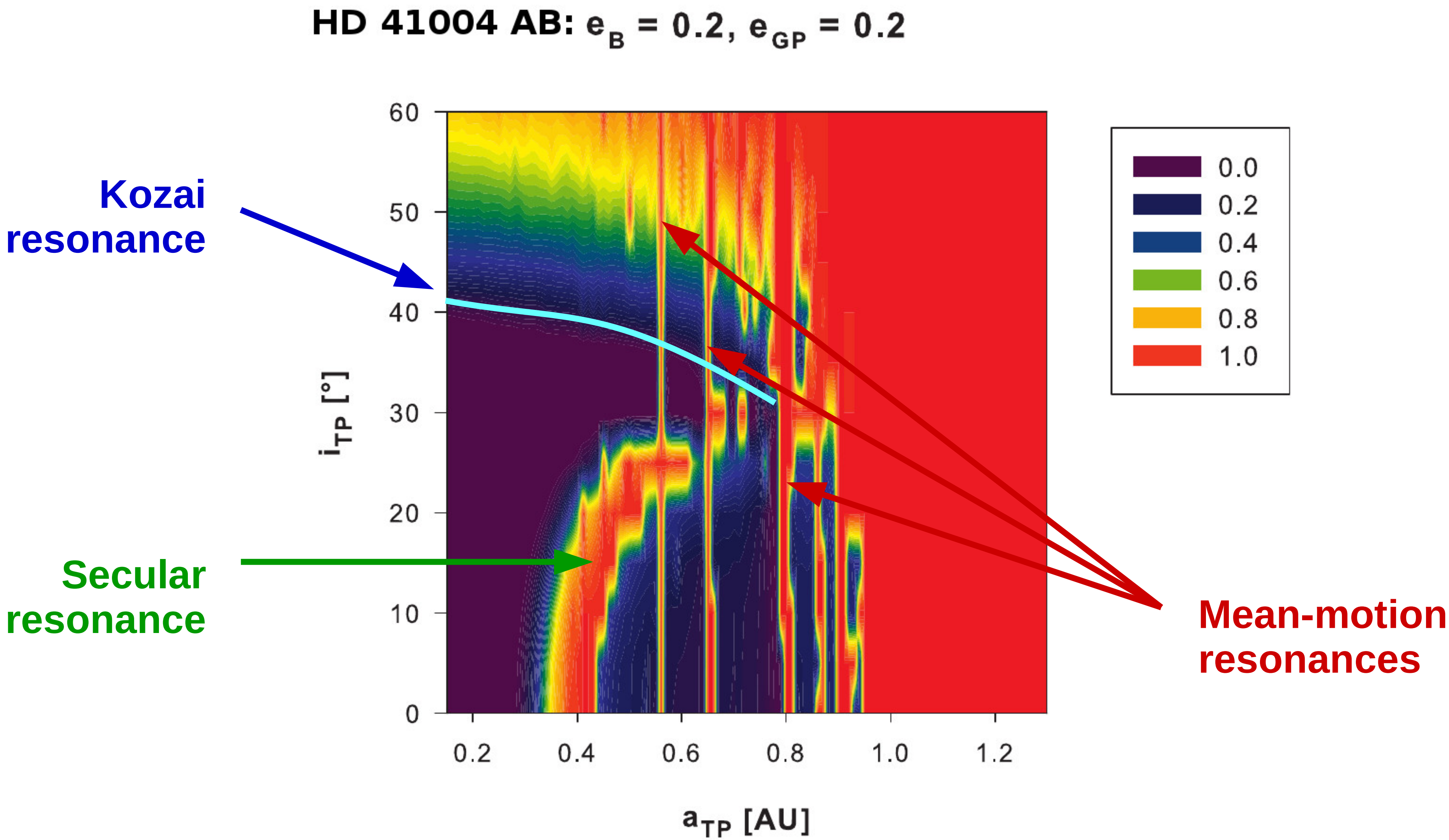
1. Motivation
2. Fourier transforms and frequency analysis
3. SigSpec
4. Method for determination of SR
5. Analytical estimates
6. Relativistic correction
7. Application to real binary star systems

1. Motivation



Pilat-Lohinger,+ (2016)

Motivation


HD 41004 AB: e_B = 0.2, e_{GP} = 0.2
Kozai resonance
Secular resonance
Mean-motion resonances
60
50
40
30
20
10
0
i_{TP} [°]
0.2
0.4
0.6
0.8
1.0
1.2
a_{TP} [AU]
0.0
0.2
0.4
0.6
0.8
1.0

Motivation – secular theory

- General secular solution for a test particle (TP) in (*h*,*k*) variables
- Proper frequency g of TP
- Small divisor for $g - g_i \approx 0$
- Need to know secular eigenfrequencies g_i of perturbers

$$h(t) = e_{\rm free} \sin(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \sin(g_i t + \varphi_i)$$

$$k(t) = e_{\rm free} \cos(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \cos(g_i t + \varphi_i)$$

$$g = \frac{1}{4} n \sum_{j=1}^{N} \frac{m_j}{M} \alpha_j^2 \, b_{3/2}^{(1)}(\alpha_j)$$

Motivation – secular theory

- Calculate g_i with Laplace-Lagrange theory
- Dependence of g_i on **eccentricity** is ignored
- Various analytical estimates (see later)

$$\dot{\mathbf{h}} = \mathbf{A}\mathbf{k}$$

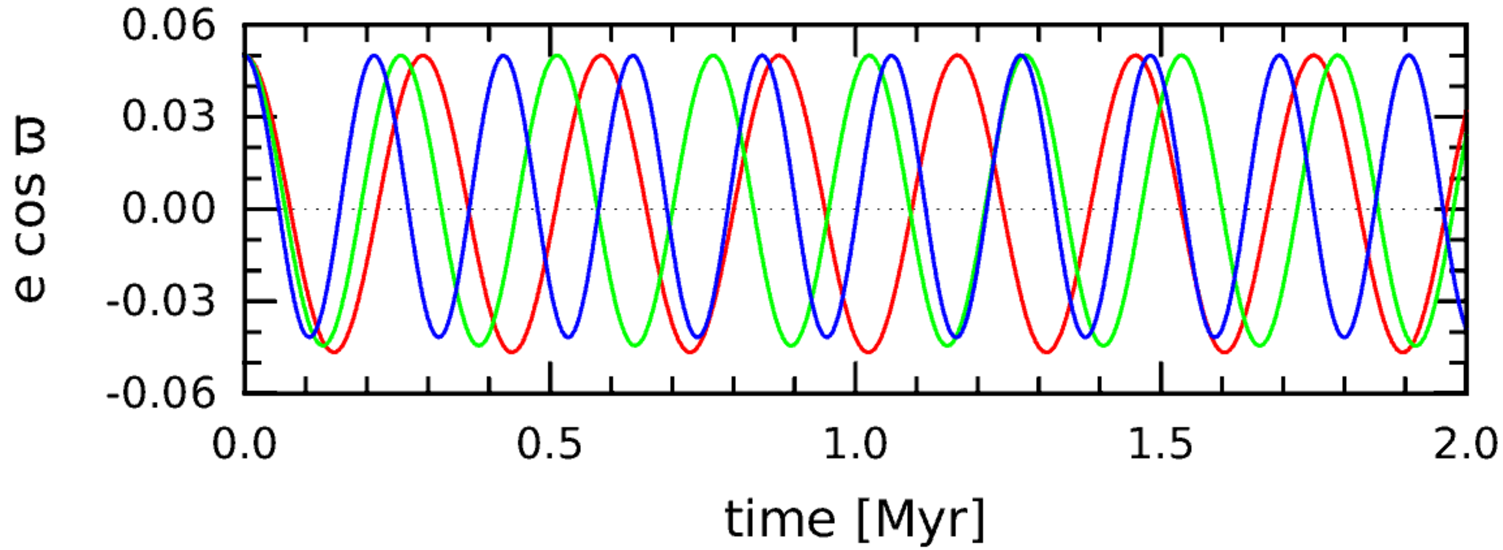
$$\dot{\mathbf{k}} = -\mathbf{A}\mathbf{h}$$

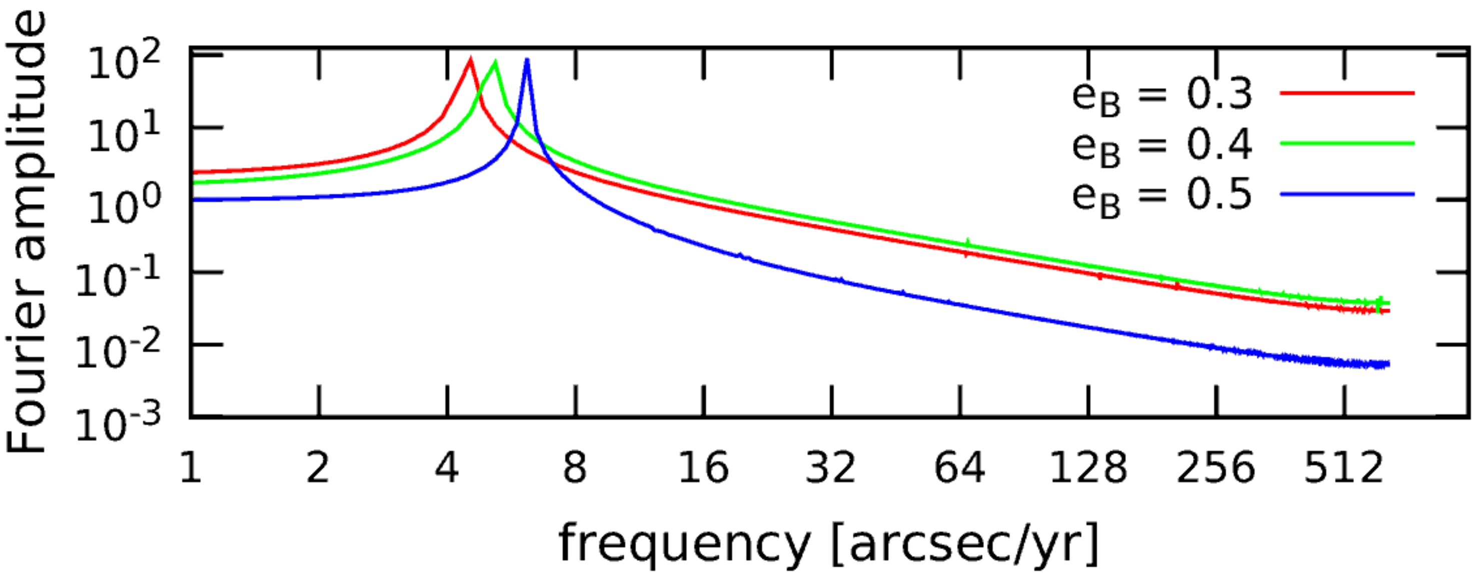
$$A_{j,j} = +\frac{1}{4} n_j \sum_{k=1}^{N} \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(1)}(\alpha_{j,k})$$

$$A_{j,k} = -\frac{1}{4} n_j \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(2)}(\alpha_{j,k})$$

$$\det(\mathbf{A} - g\mathbf{1}) = 0$$

Motivation – frequency vs eccentricity

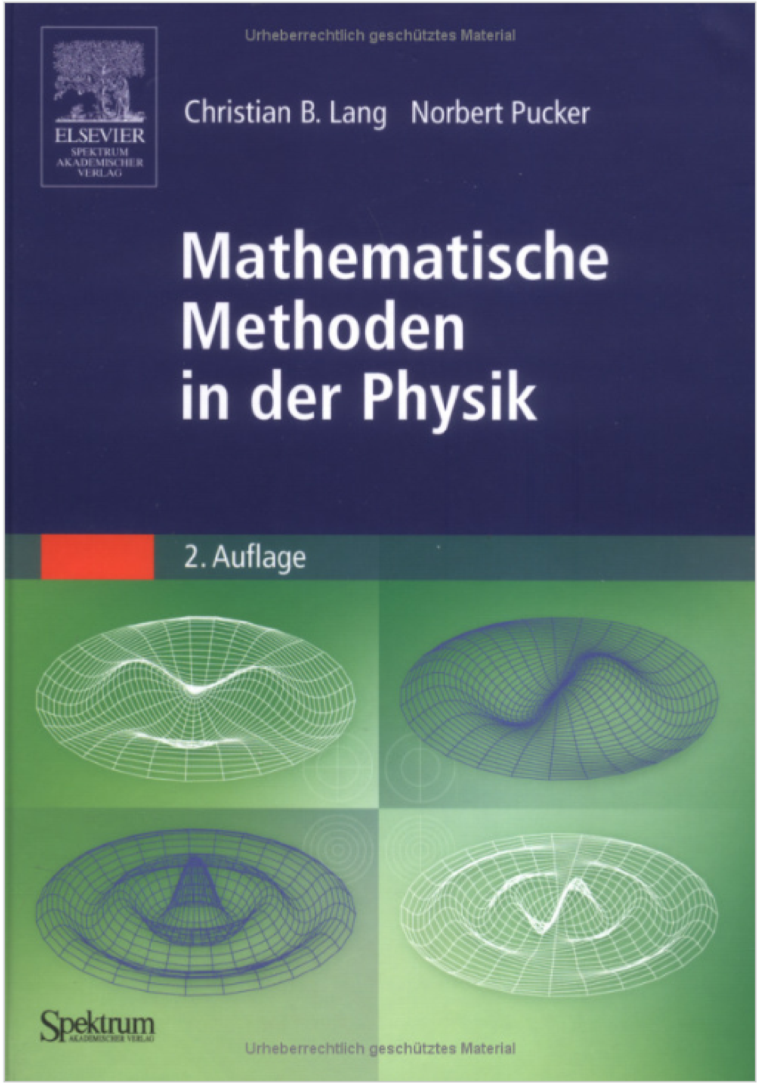


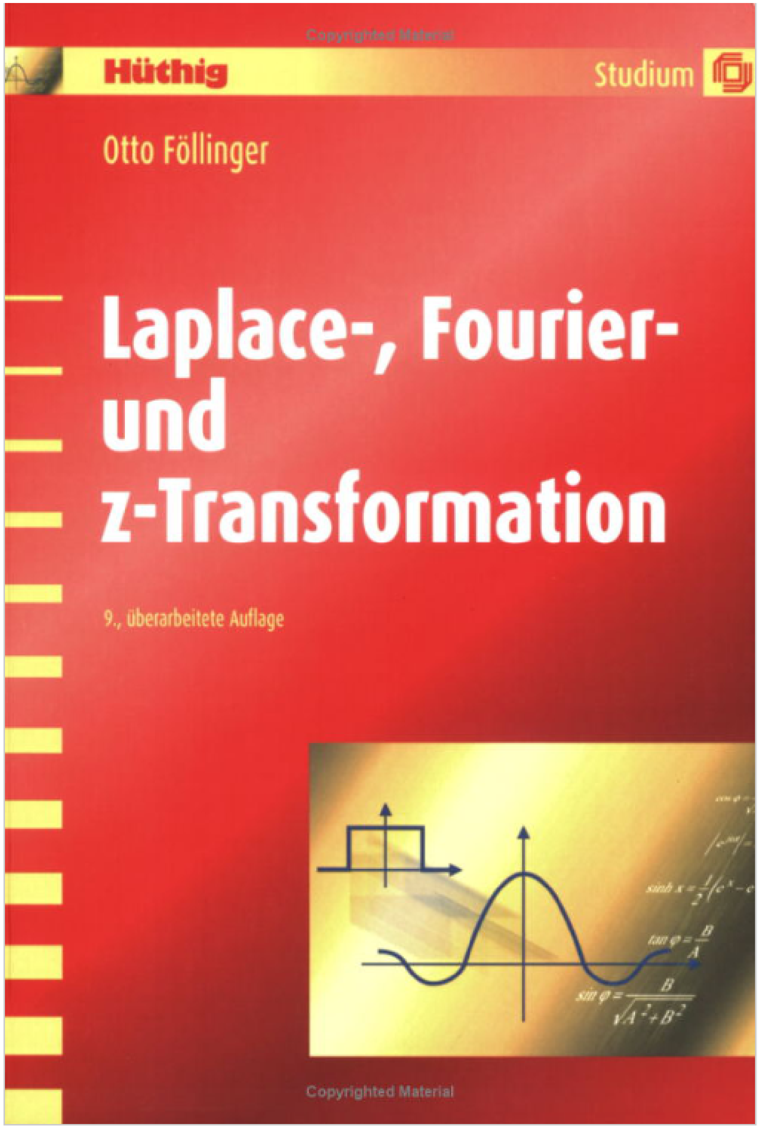


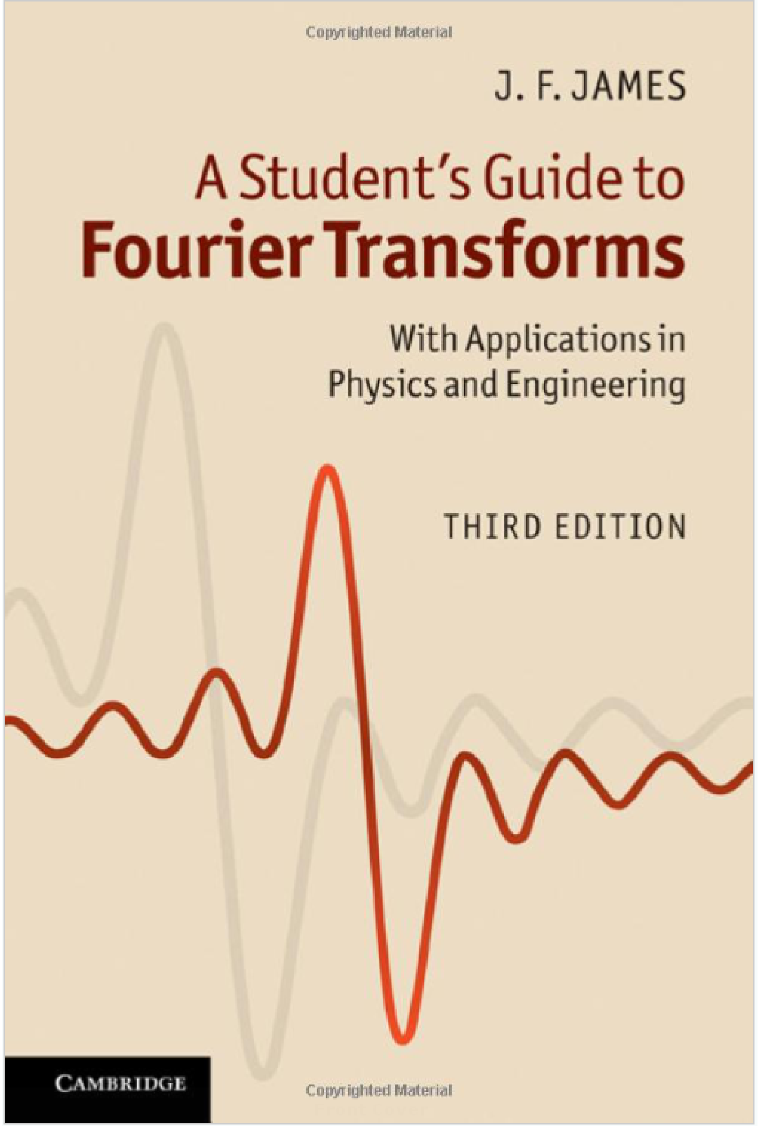
Motivation – method

- Semi-analytical method for multi-planet systems of binary stars … Pilat-Lohinger,+ (2016), Bazsó,+ (2017)
- Numerical part (full 3-body problem):
  - Single numerical integration of binary star – giant planet system
  - Frequency analysis for giant planet's frequency g_{GP}
- Analytical part (restricted 4-body problem):
  - Perturbations from secondary star + giant planet
  - Laplace-Lagrange theory for test particle proper frequency g_{TP}
- Combination of “best” methods

2. Fourier transforms

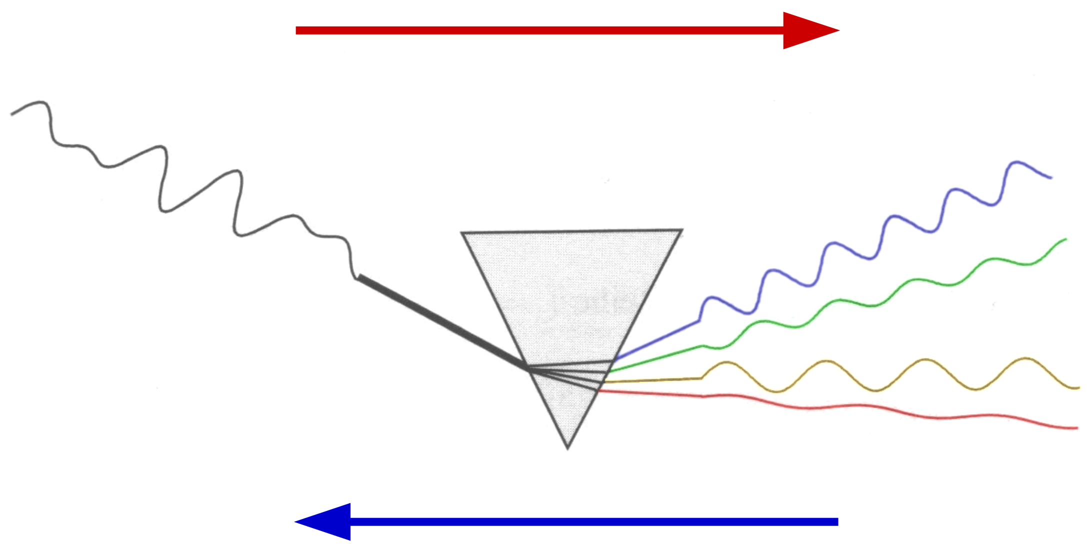




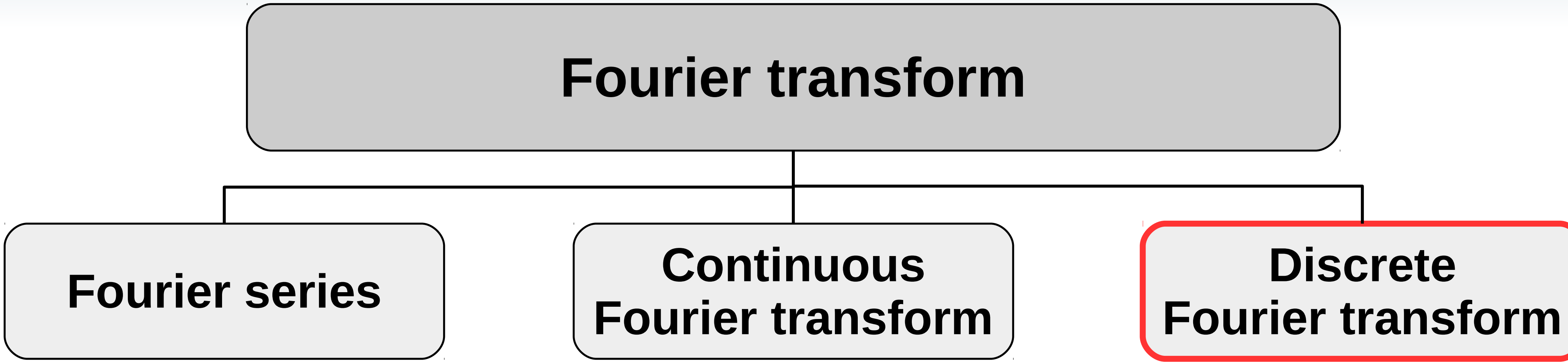


Fourier transforms – principle

- Analogy = light + prism
- Mapping of function/data ↔ frequency spectrum
- **Fourier analysis:** decompose signal into basic frequencies
- **Fourier synthesis:** assemble signal from frequency spectrum



Fourier transforms – types



Fourier series	Continuous Fourier transform	Discrete Fourier transform
Continuous interval + periodic	Continuous interval (ℝ) + non-periodic	Discrete interval + periodic
Discrete frequency spectrum	Continuous frequency spectrum	Discrete frequency spectrum

Discrete Fourier Transform (DFT)

- Sampling a continuous function $y(x)$ at N discrete values $k = 0 \ldots N\text{-}1$
- DFT: $\{y_k\} \in \mathbb{C} \rightarrow \{c_n\} \in \mathbb{C}$
- Fourier coefficients c_n
- Inverse DFT:

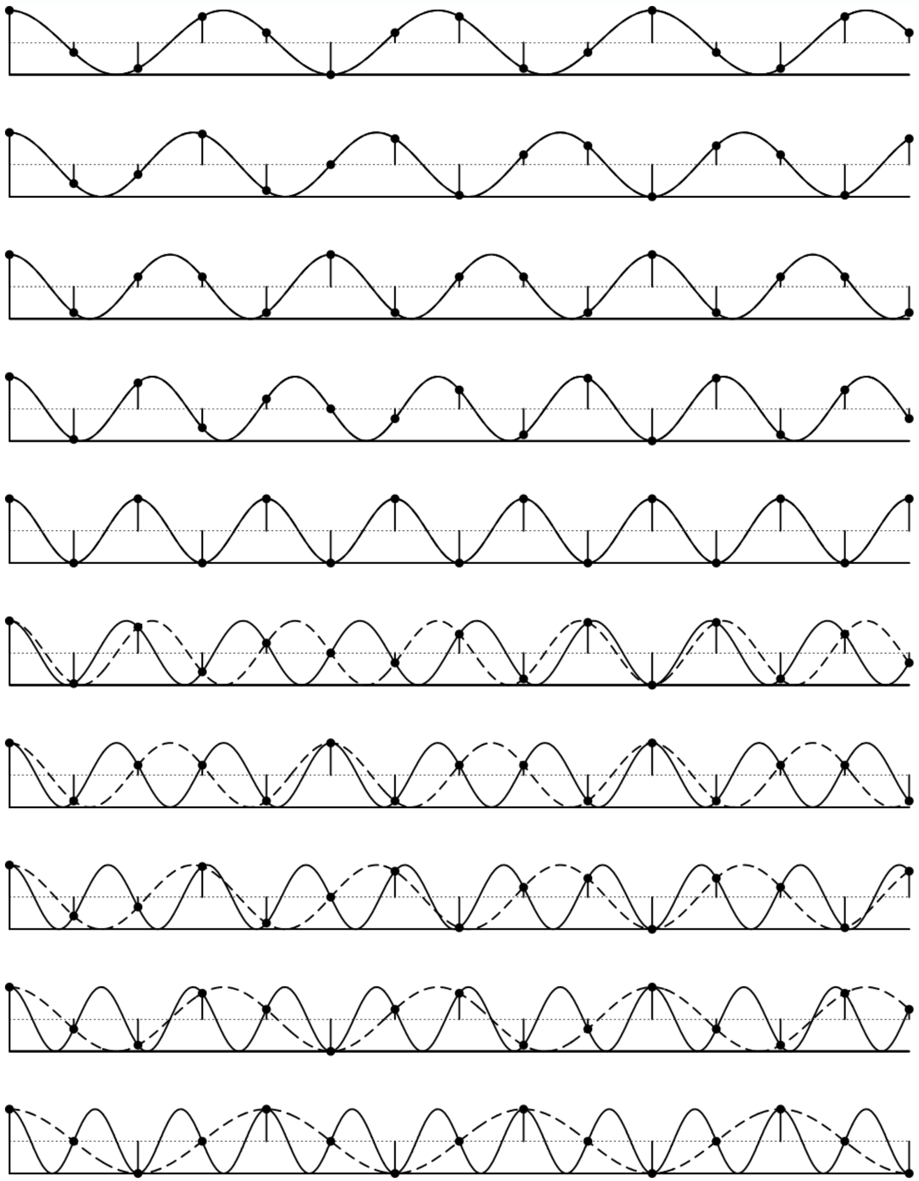
$$y_k = y(x_k), \quad x_k = 2\pi k/N$$

$$c_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k z^{kn}, \quad z = \exp(-i2\pi/N)$$

$$y_j = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n z^{-nj}$$

DFT artefacts – aliasing

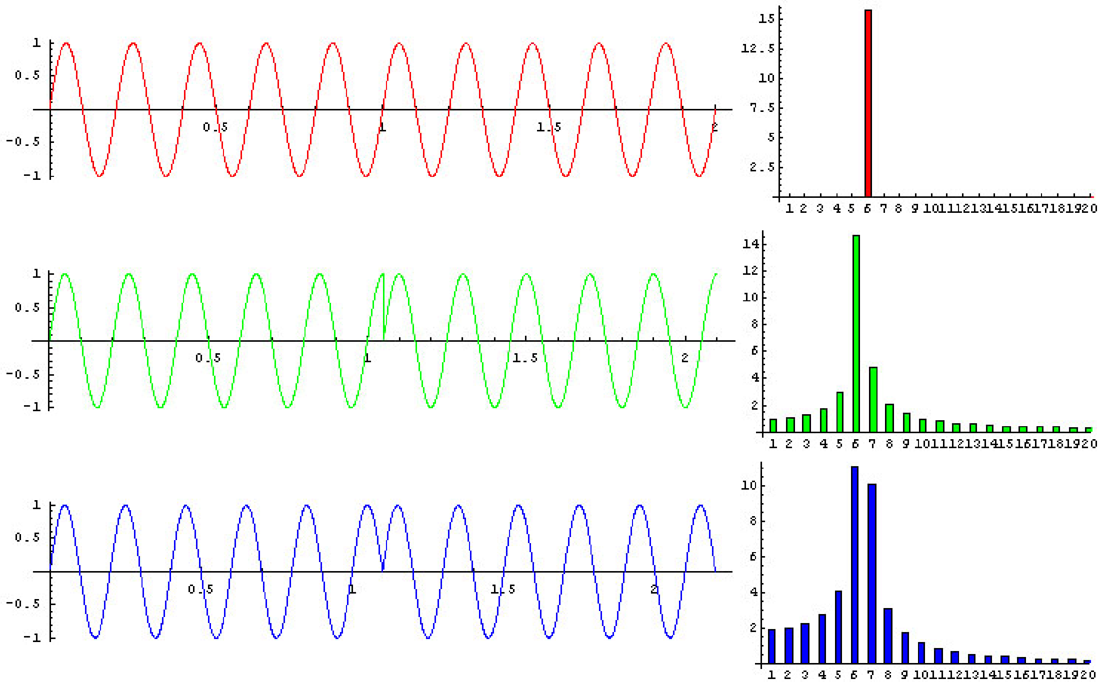
- Sampling rate of continuous function
- Nyquist frequency $f_{\text{Nyq}} > 2\ f_{\text{max}}$
- Aliasing effect by introducing “fake” low frequencies



http://wikipedia.org

DFT artefacts – leakage

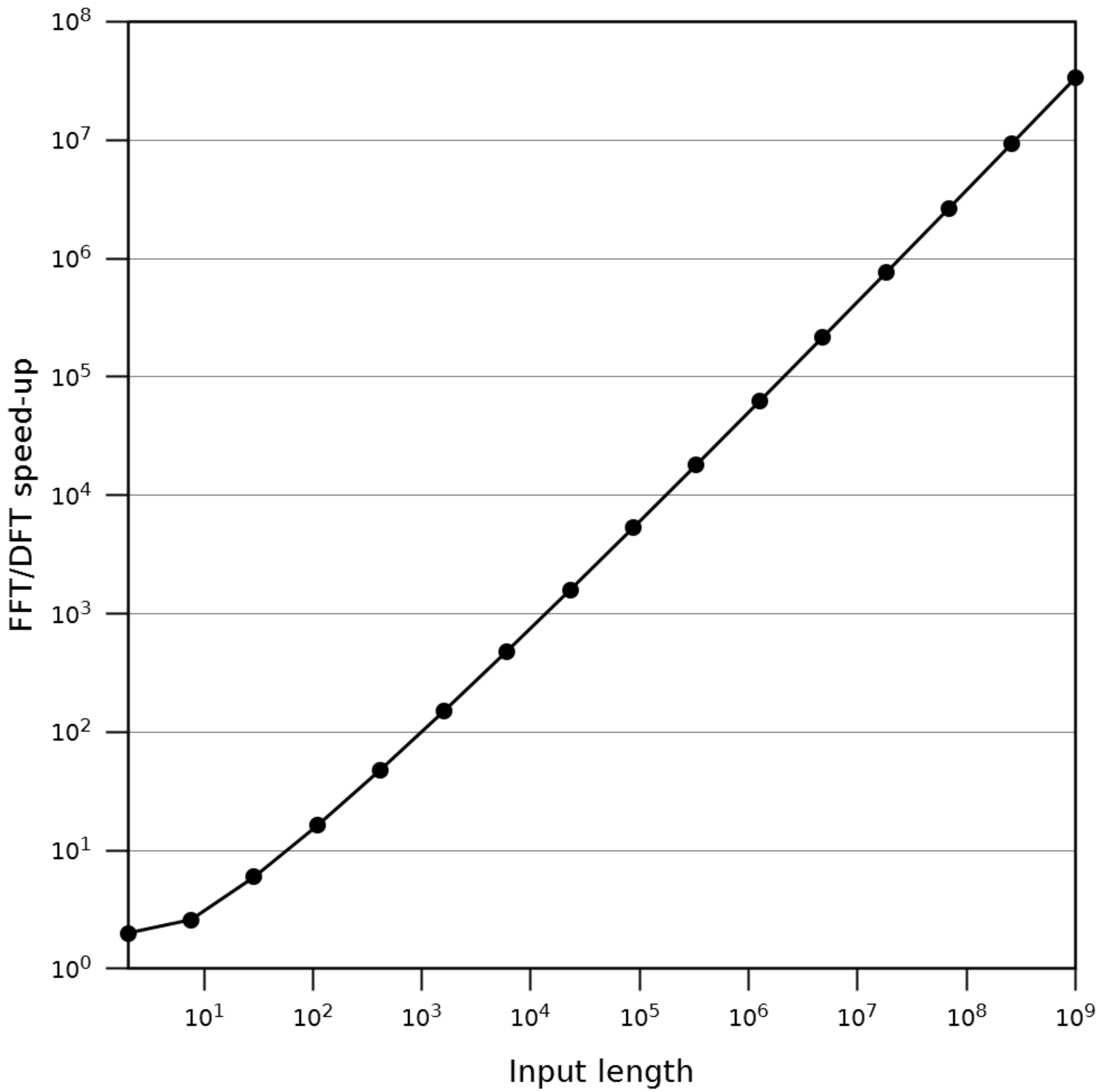
- Sampling a periodic signal
- Sampling interval not an integer multiple of signal period
- Discrete spectrum with nearby “ghost” frequencies



http://wikipedia.org

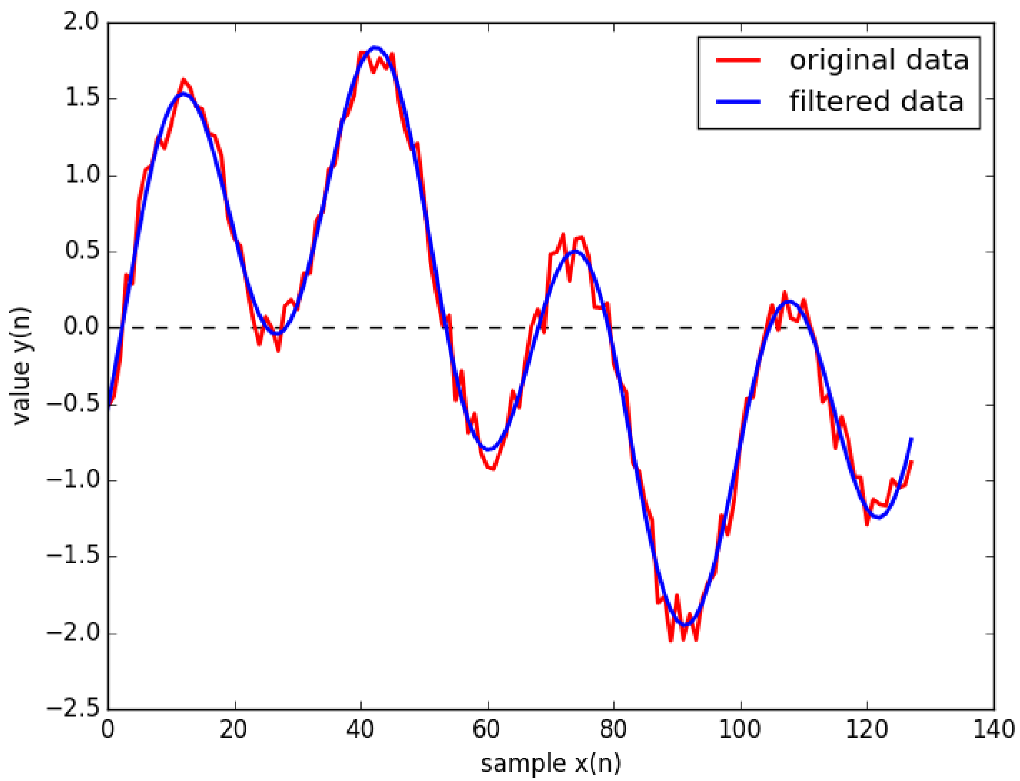
DFT and FFT

- DFT ~ $O(N^2)$
- Fast Fourier Transform
  FFT ~ $O(N \log_2 N)$
  - Cooley & Tukey (1965)
  - Frigo & Johnson (2005), http://www.fftw.org



DFT – demonstration

- DFT of signal with 128 samples + noise
- Magnitude of Fourier coefficients c_n
- “Amplitude filter”
- Reconstructing signal



Frequency analysis

- Robutel & Laskar (2001)
- Analyse time series *z*(*t*)
- Quasi-periodic decomposition of signal over time interval [0, *T*]
- Determine frequency for first/second half of data
- Compare relative shift of frequency in half intervals

$$z(t) = a(t)\exp(i\lambda(t))$$

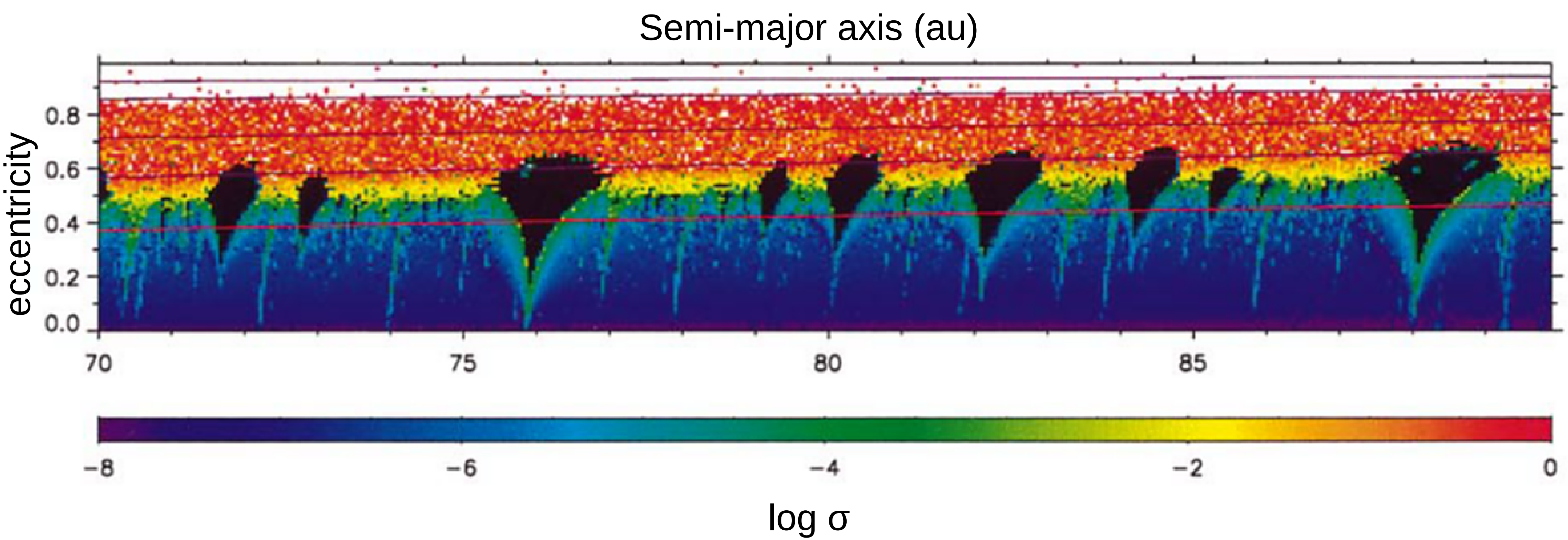
$$\tilde{z}(t) = a_0 \exp(i n_0 t) + \sum_{j=1}^{N} a_j \exp(i n_j t)$$

$$\sigma = 1 - n^{(2)}/n^{(1)}$$

$$n^{(1)} \in [0, T/2]$$

$$n^{(2)} \in [T/2, T]$$

Frequency map



Robutel & Laskar (2001)

3. SigSpec

```
 SSSSSS  ii          SSSSSS
SS    SS            SS    SS
SS       ii  gggg g SS       p pppp   eeeee   cccccc
SS       ii gg   gg SS       pp   pp ee   ee cc   cc
 SSSSSS  ii gg   gg  SSSSSS  pp   pp ee   ee cc
      SS ii gg   gg       SS pp   pp eeeeeee cc
      SS ii gg   gg       SS pp   pp ee      cc
SS    SS ii gg   gg SS    SS pp   pp ee   ee cc   cc
 SSSSSS  ii  gggggg  SSSSSS  pppppp   eeeee   ccccc
                 gg          pp
            gg   gg          pp
             ggggg           pp
```

SigSpec

- Reegen (2007):
  - Frequency- and phase-resolved “significance” in Fourier space
  - Identify frequencies in time-series
  - Eliminate peaks due to noise
- Documentation:
  http://homepage.univie.ac.at/peter.reegen/manual/node2.html
  http://esoads.eso.org/abs/2011CoAst.163....3R
- Download:
  http://homepage.univie.ac.at/peter.reegen/download.html

SigSpec – method

- DFT of time series with arbitrary sampling (non-equidistant, gaps, …)
- **Sig**nificance **Spec**trum for DFT amplitude spectrum
- Compute probability density function for each amplitude *A*
- False alarm probability $\Phi_{FA}(A)$
- Spectral significance $\mathrm{sig}(A) = -\log[\Phi_{FA}(A)]$
- Example: $\mathrm{sig}(A) = 5$ … peak generated by noise in $1{:}10^5$ cases

SigSpec – application

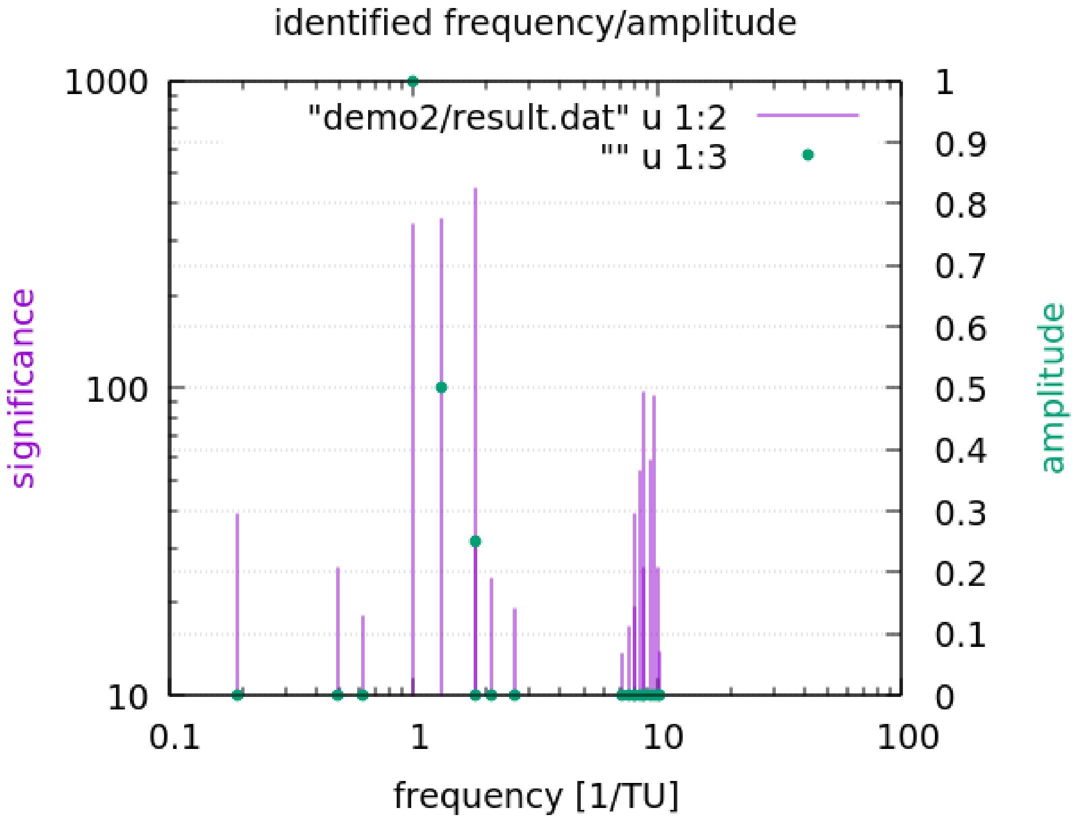
- A SigSpec project
  - Directory <project> for output
  - File <project>.dat for time series input
  - File <project>.ini for settings
- Run SigSpec program on <project>

SigSpec – demonstrations

- **Demo #1:**
  superposition of 3
  sine functions ~ sin($f_i\, x$)

- **Demo #2:**
  sine functions ~ sin($2\pi\, f_i\, x$)

- **Demo #3:**
  sine functions + random noise



4. Determination of secular resonances

Semi-analytical method

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- Numerical part (full 3-body problem):
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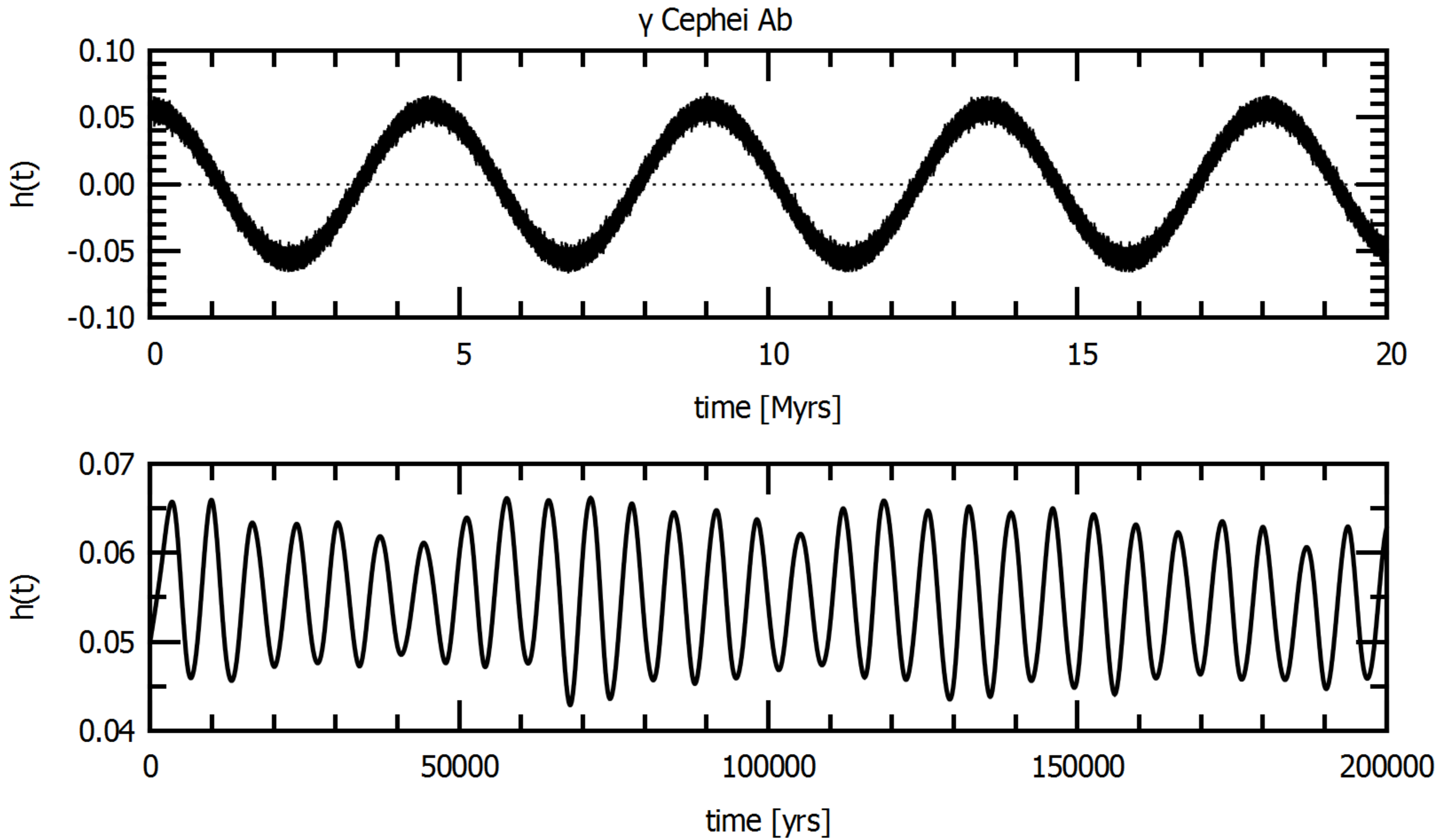
Method – numerical integration

- Use *N*-body integrator (Lie series, Mercury, …)
- Convert output to Laplace-Lagrange variables $\{h,k,p,q\}$

$$h = e\sin(\omega + \Omega) \qquad p = \sin(i/2)\sin\Omega$$

$$k = e\cos(\omega + \Omega) \qquad q = \sin(i/2)\cos\Omega$$

Method – numerical integration

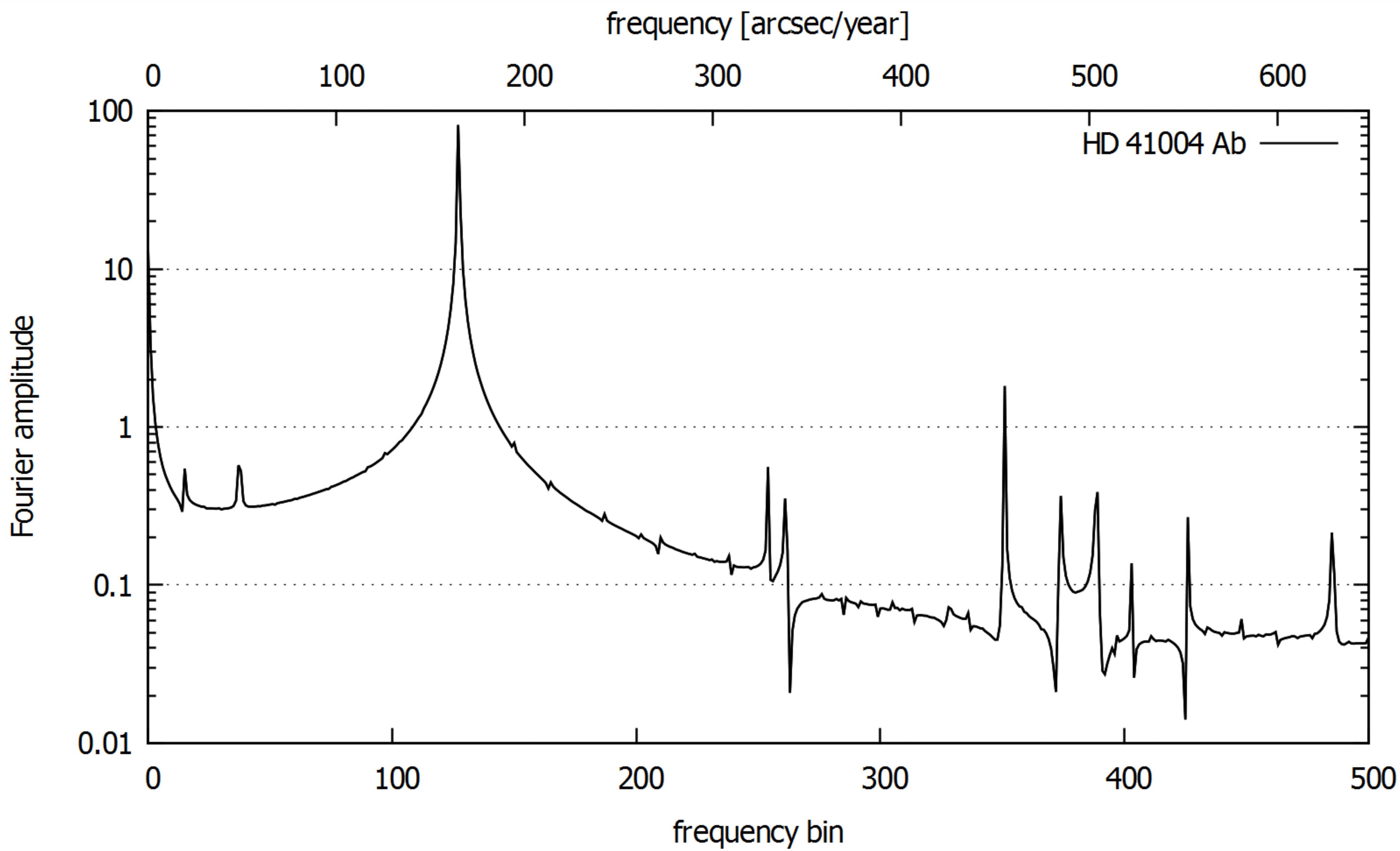


Method – frequency analysis

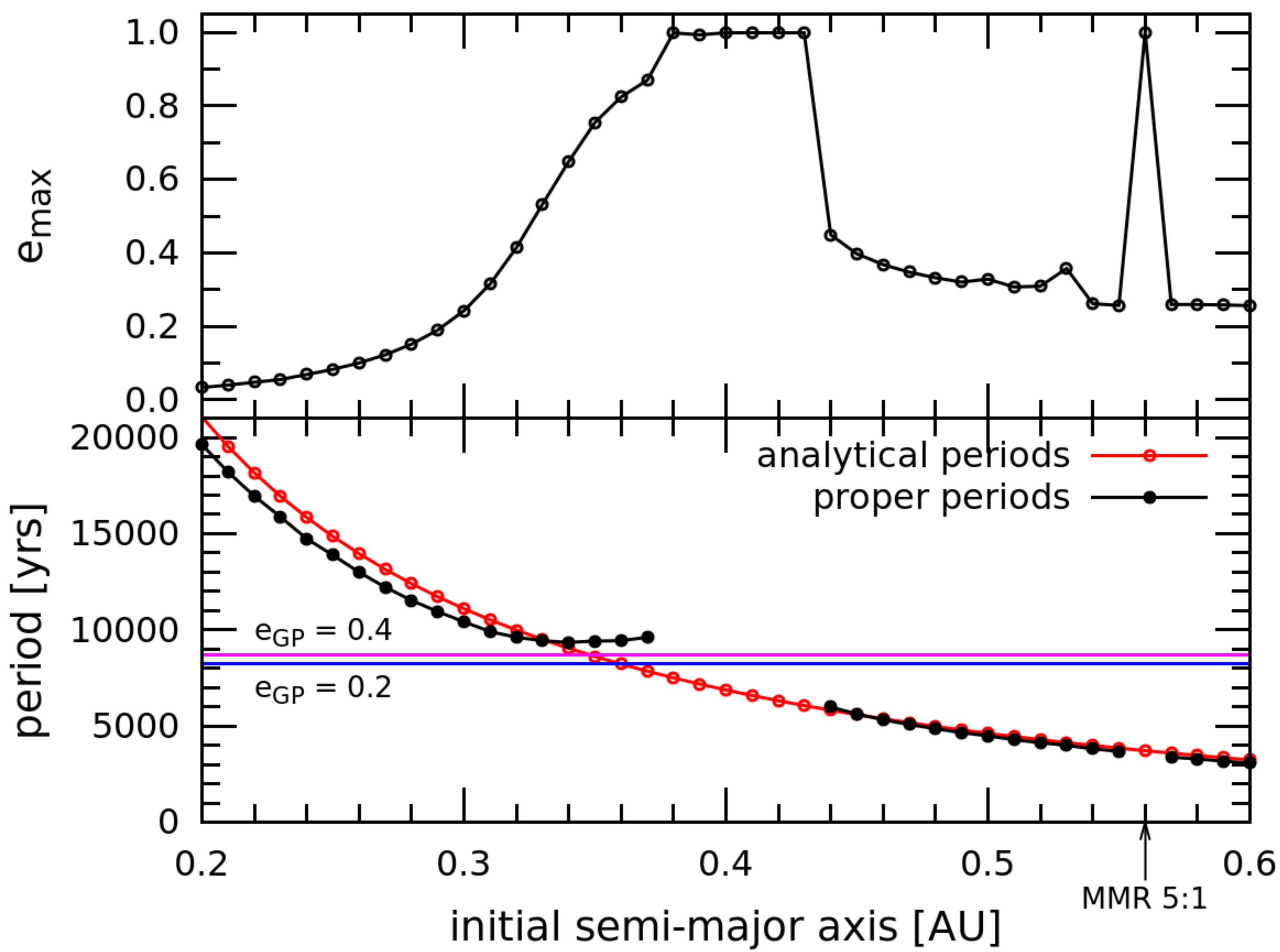
- Time series for $h(t)$, $k(t)$
- Dynamical spectrum of time series
- Largest amplitudes are precession frequencies for giant planet (GP) and secondary star

$$\langle g_{GP} \rangle = \frac{1}{T} \int_0^T g_{GP}(t)\,\mathrm{d}t$$

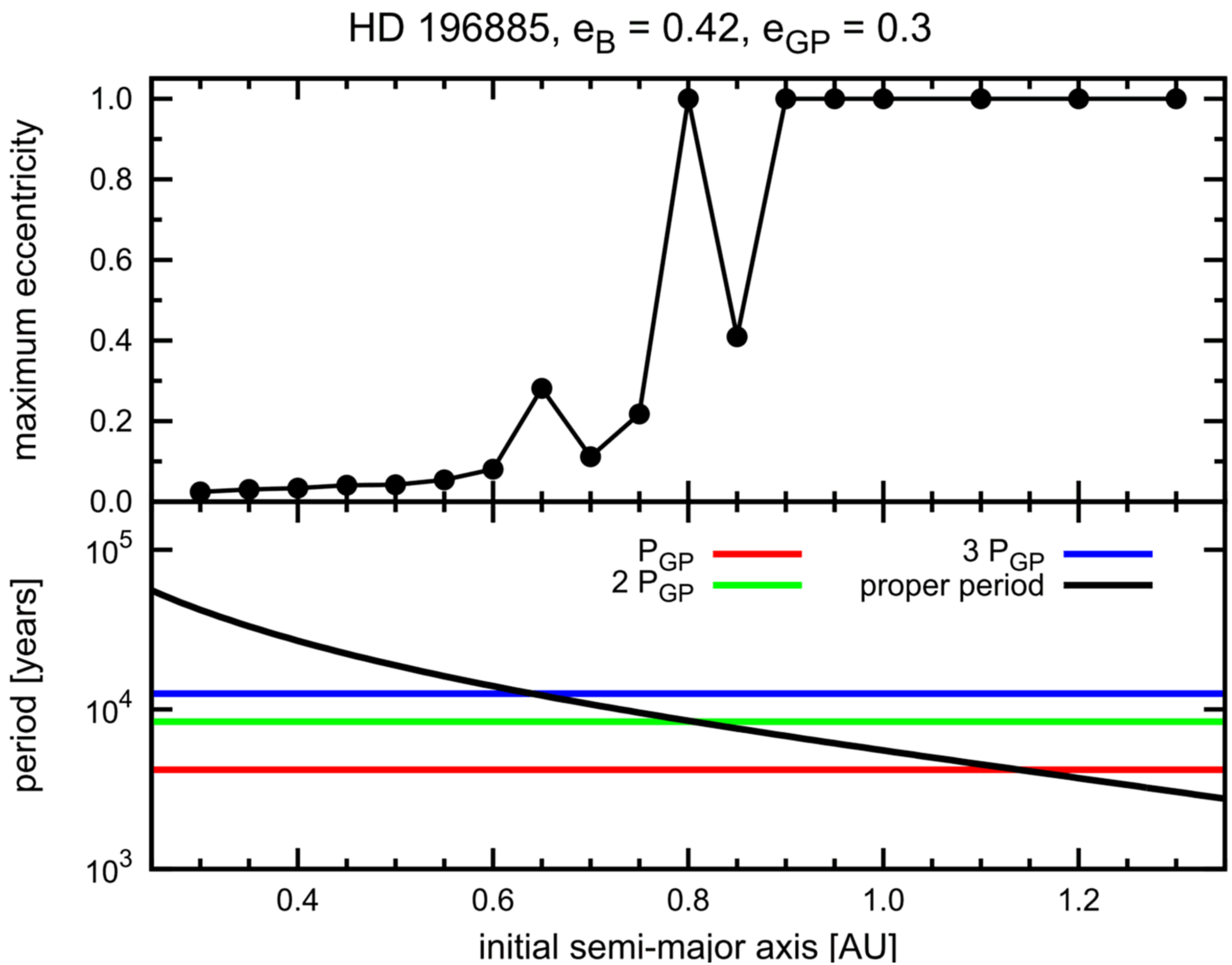
Method – frequency analysis



Method – analytical part
Linear secular resonance



Method – analytical part
Non-linear secular resonances



Summary

- Accurate detection of SR location needs accurate g_i
- Laplace-Lagrange theory limited

- Use numerical integration + frequency analysis
- Combine with Laplace-Lagrange theory for test particles (initially circular)

References

- Bazsó, Pilat-Lohinger, Eggl, Funk, Bancelin, Rau (2017), MNRAS 466, 1555–1566
- Cooley, Tukey (1965), Math. Comput. 19, 297–301
- Frigo, Johnson (2005), Proc. IEEE 93, 216–231
- Pilat-Lohinger, Bazsó, Funk (2016), AJ 152, 139
- Reegen (2007), A&A 467, 1353–1371
- Robutel, Laskar (2001), Icarus 152, 4–28