The Effect of Resonances

Part 2

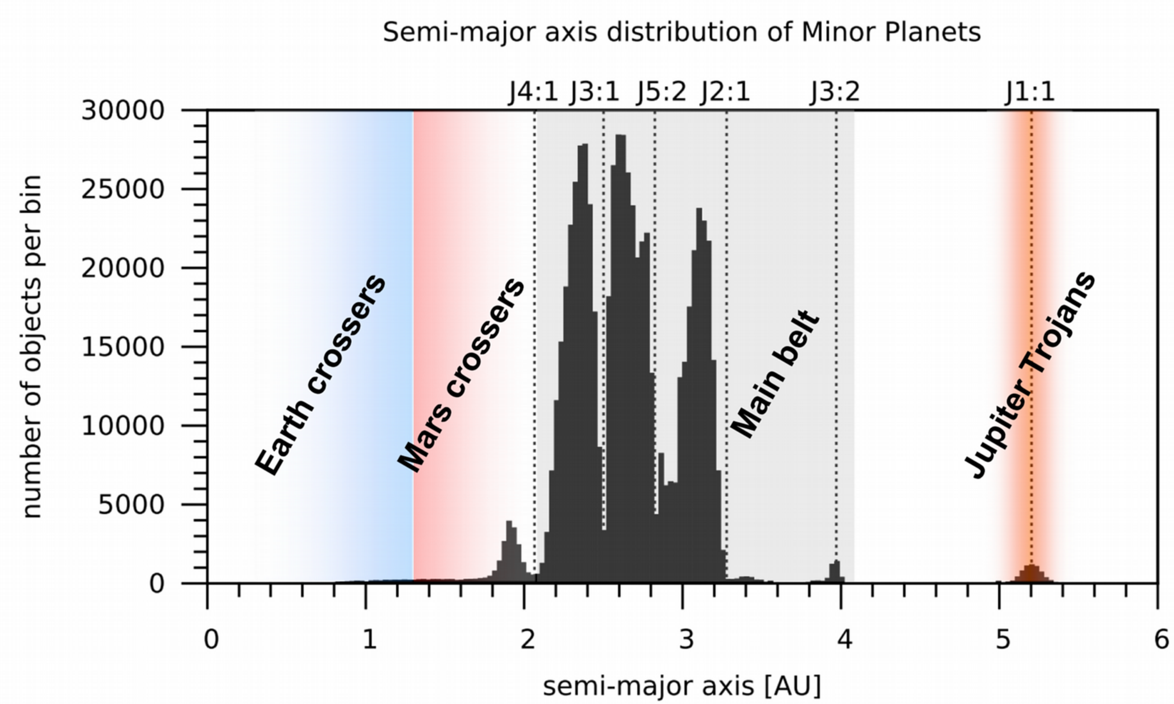
Topics overview

1. Definition and examples of “resonances”
2. Disturbing function
3. Mean-motion resonance (MMR)
4. Secular resonance (SR) – continued
5. Kozai-Lidov resonance
6. Evection resonance
7. Other resonances

3. Mean-motion resonance – recap

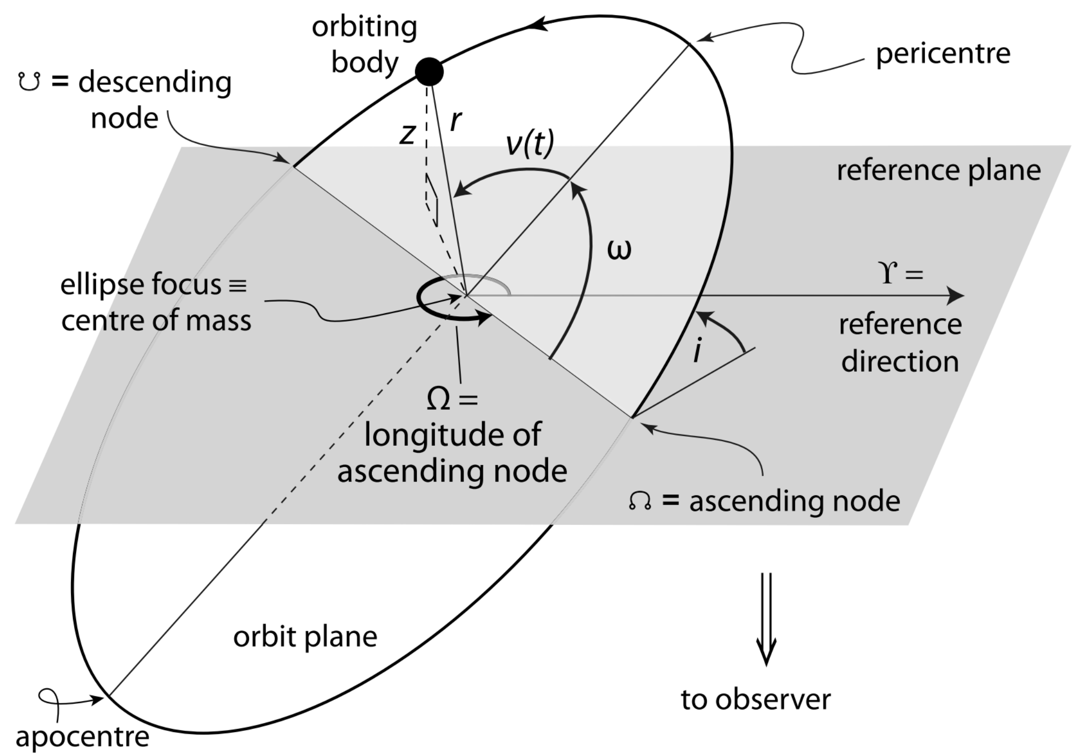
- MMR → resonance between two orbital frequencies
  $q\, n_1 - p\, n_2 = 0$
- Critical angle → small divisor
- Resonance location: simple formula

$$a_{\rm res} = a' \left(\frac{n'}{n}\right)^{2/3} \left(\frac{M+m}{M+m'}\right)^{1/3}$$



4. Secular resonance – recap

- SR → resonance between two orbital precession frequencies
- Precession of line of apsides → freq. g
- Precession of line of nodes → freq. s
- Time-scale $T_{sec} >> T_{rev}$



Perryman (2011)

SR – secular variables

- Laplace-Lagrange variables
- Decoupling of eccentricity / inclination (to lowest order) in averaged disturbing function

$$h = e \sin(\omega + \Omega) \qquad p = \sin(i/2) \sin \Omega$$
$$k = e \cos(\omega + \Omega) \qquad q = \sin(i/2) \cos \Omega$$

SR – solutions for massive bodies

- Equations of motion in variables (h,k) = system of linear differential equations
- Secular eigenfrequencies = eigenvalues g_i of matrix **A**
- Laplace coefficients $b_n^{(k)}(\alpha)$

$$\dot{\mathbf{h}} = \mathbf{A}\mathbf{k}$$

$$\dot{\mathbf{k}} = -\mathbf{A}\mathbf{h}$$

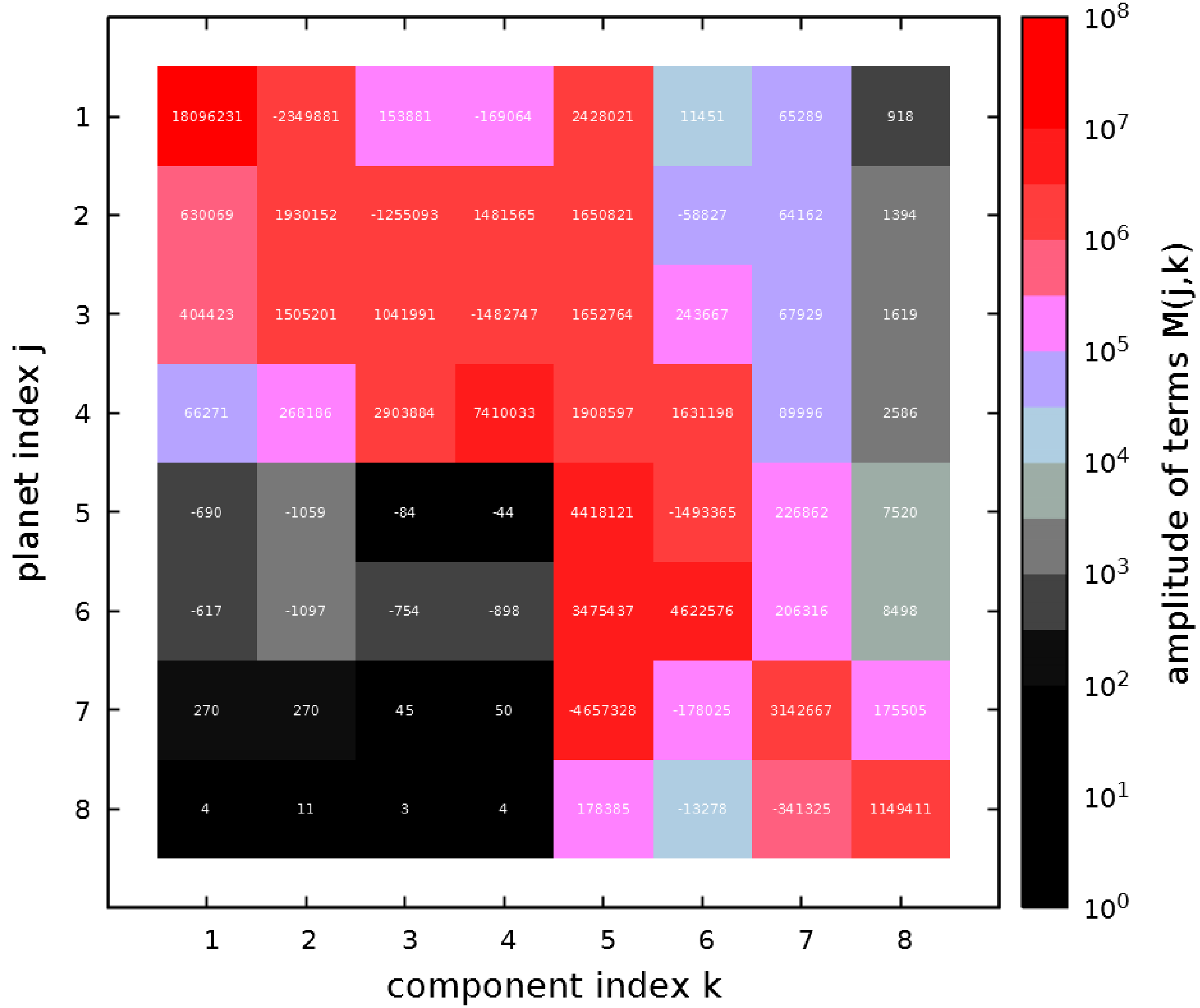
$$A_{j,j} = +\frac{1}{4} n_j \sum_{k=1}^{N} \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(1)}(\alpha_{j,k})$$

$$A_{j,k} = -\frac{1}{4} n_j \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(2)}(\alpha_{j,k})$$

$$\det(\mathbf{A} - g\mathbf{1}) = 0$$

SR – solutions for massive bodies

Matrix of eigenvectors to eigenvalues for matrix **A**



SR – solutions for massive bodies

Python notebook for demonstration

- Example 1: Outer Solar System
- Example 2: Gamma Cephei
- Example 3: Jupiter in a “wide” binary star system

SR – test particle

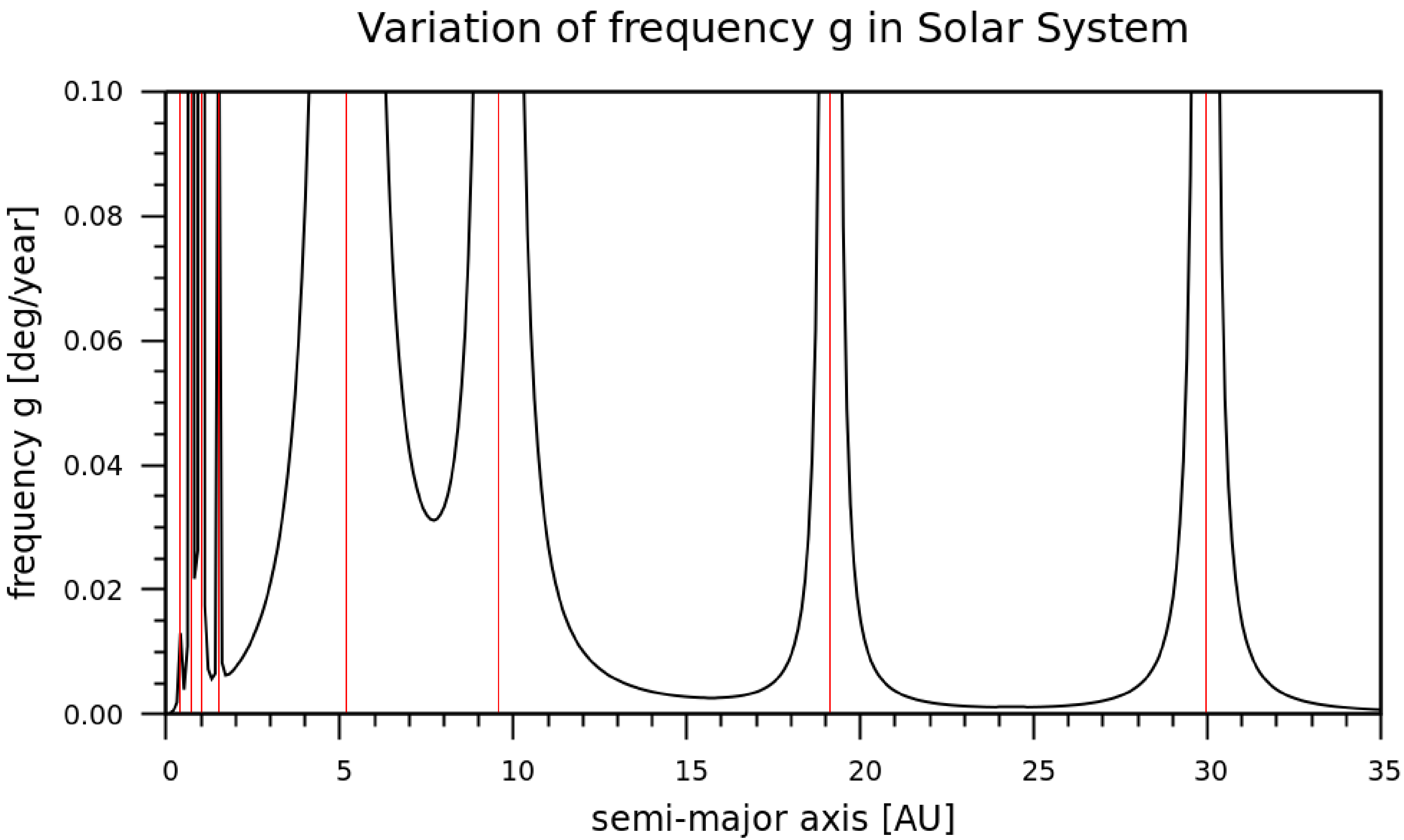
- Disturbing function for a TP with *N* massive perturbers
- Proper frequency *g* of TP
- General solution for TP in (*h*,*k*) variables
- Small divisor for $g - g_i \approx 0$
- Proper (free) + forced eccentricity / inclination

$$\mathcal{R} = n\,a^2 \left[\frac{1}{2} g \left(h^2 + k^2 \right) + \sum_{j=1}^{N} A_j \left(h\,h_j + k\,k_j \right) \right]$$

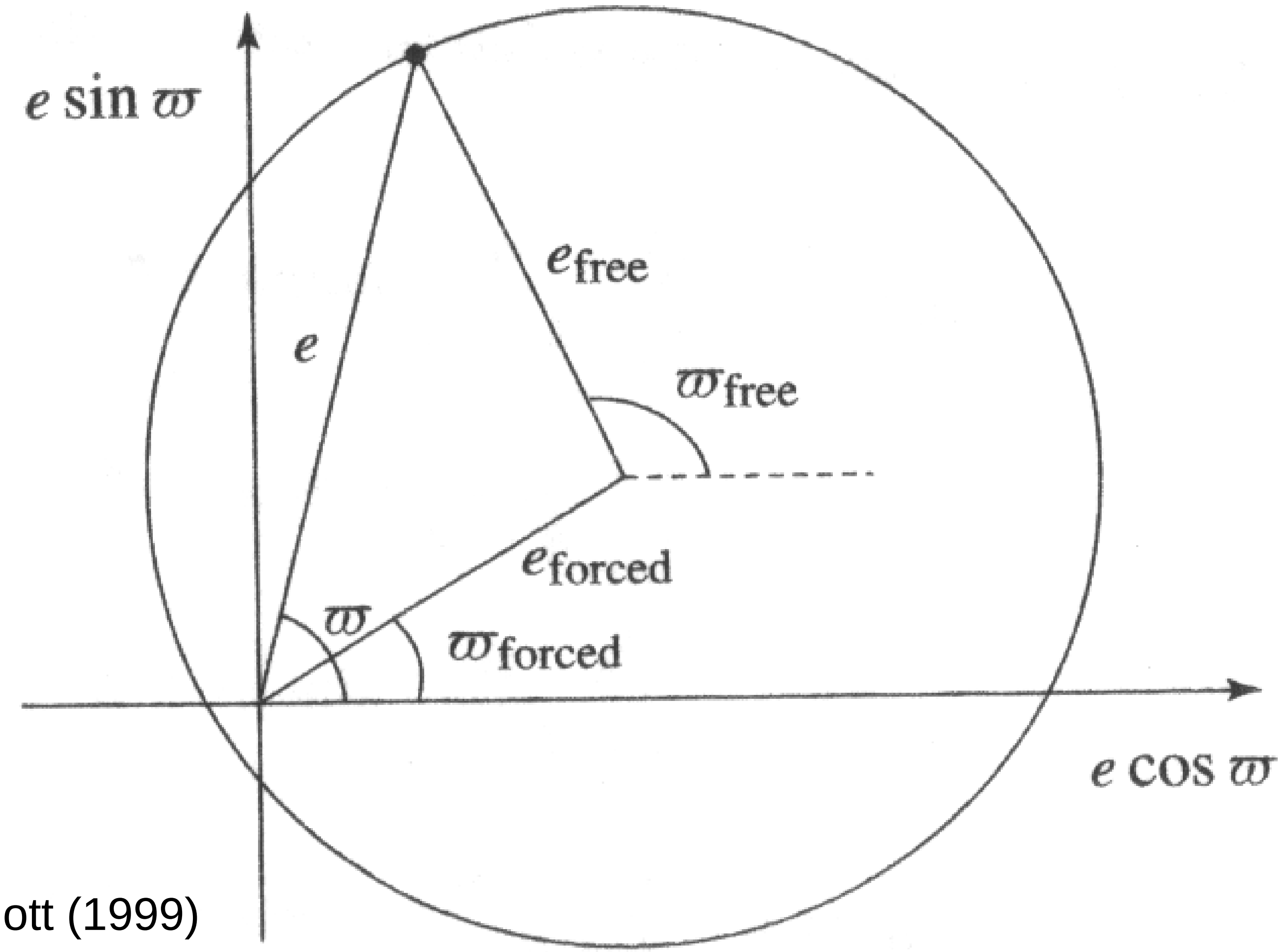
$$g = \frac{1}{4} n \sum_{j=1}^{N} \frac{m_j}{M} \alpha_j^2 \, b_{3/2}^{(1)}(\alpha_j)$$

$$\begin{aligned} h(t) &= e_{\rm free} \sin(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \sin(g_i t + \varphi_i) \\ &= h_{\rm free}(t) + h_0(t) \\ k(t) &= e_{\rm free} \cos(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \cos(g_i t + \varphi_i) \\ &= k_{\rm free}(t) + k_0(t) \\ e_{\rm forced} &= \sqrt{h_0^2 + k_0^2} \\ i_{\rm forced} &= \sqrt{p_0^2 + q_0^2} \end{aligned}$$

SR – proper frequency *g*

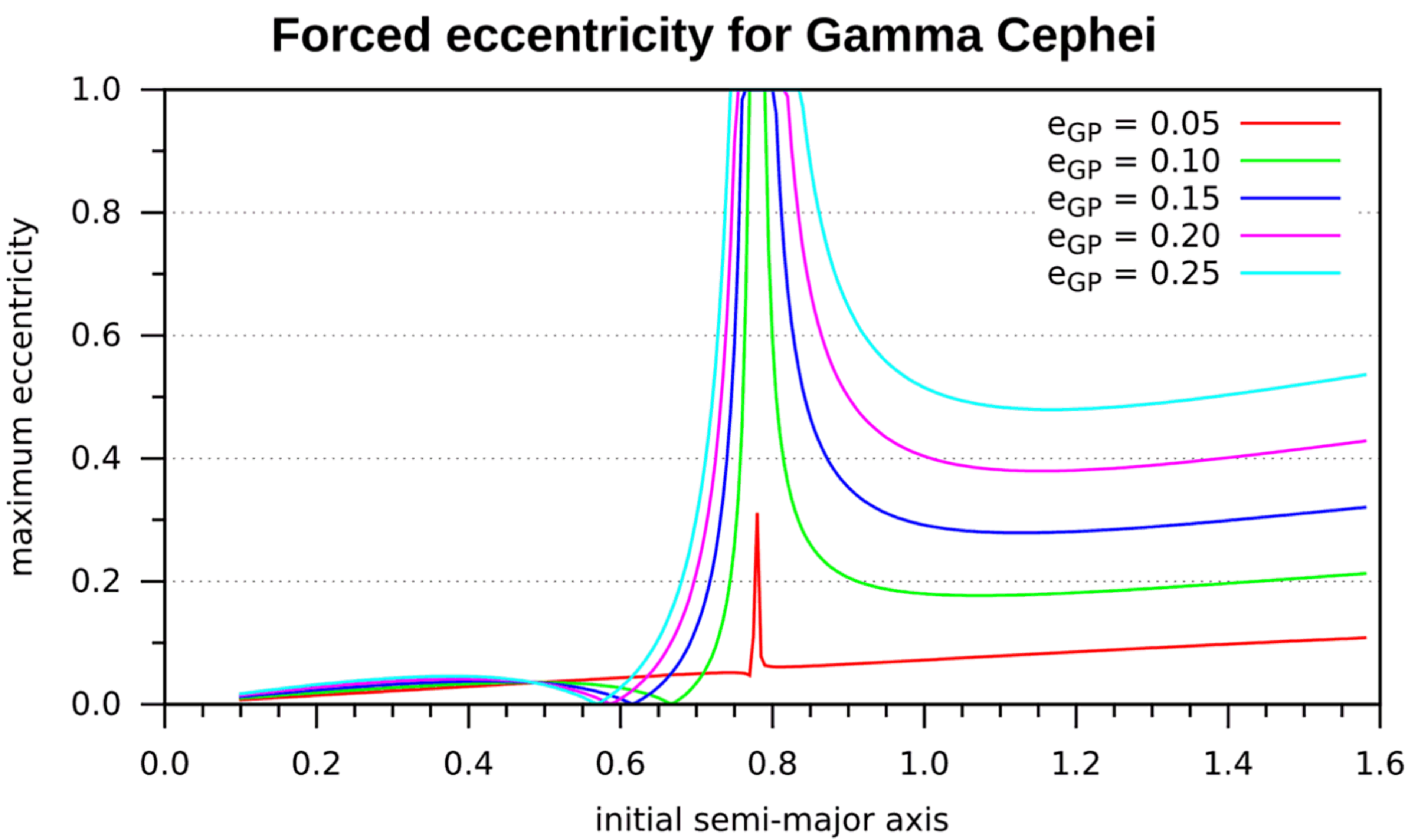


SR – free / forced eccentricity



Murray & Dermott (1999)

SR – forced eccentricity



5. Kozai-Lidov resonance (KL)

KL – dynamical setting

- Restricted 3-body problem
- Outer perturber m' on Keplerian orbit
- Distances $a \ll a'$, masses $m \ll m'$, mutual inclination

KL – theoretical concepts

- Kozai-Lidov (KL) resonance → type of secular resonance
- Resonance between orbital precession frequencies of small body under effect of perturber(s)

$$\dot{\tilde{\omega}} - \dot{\Omega} = 0 \Rightarrow \dot{\omega} = 0$$

- Libration of argument of pericenter ω about 90° / 270°
- Coupling between eccentricity / inclination oscillations
- Periodic exchange between *e*(*t*) & *i*(*t*)

KL – a note on Delaunay variables

Definition:

$$l = M \qquad L = \sqrt{\mu a}$$

$$g = \omega \qquad G = \sqrt{\mu a}\sqrt{1-e^2} = L\sqrt{1-e^2}$$

$$h = \Omega \qquad H = \sqrt{\mu a}\sqrt{1-e^2}\cos(i) = G\cos(i)$$

$$\mu = GM^3/(M+m)^2$$

Canonical system of variables:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial L} \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial M}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial G} \qquad \frac{\mathrm{d}G}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \omega}$$

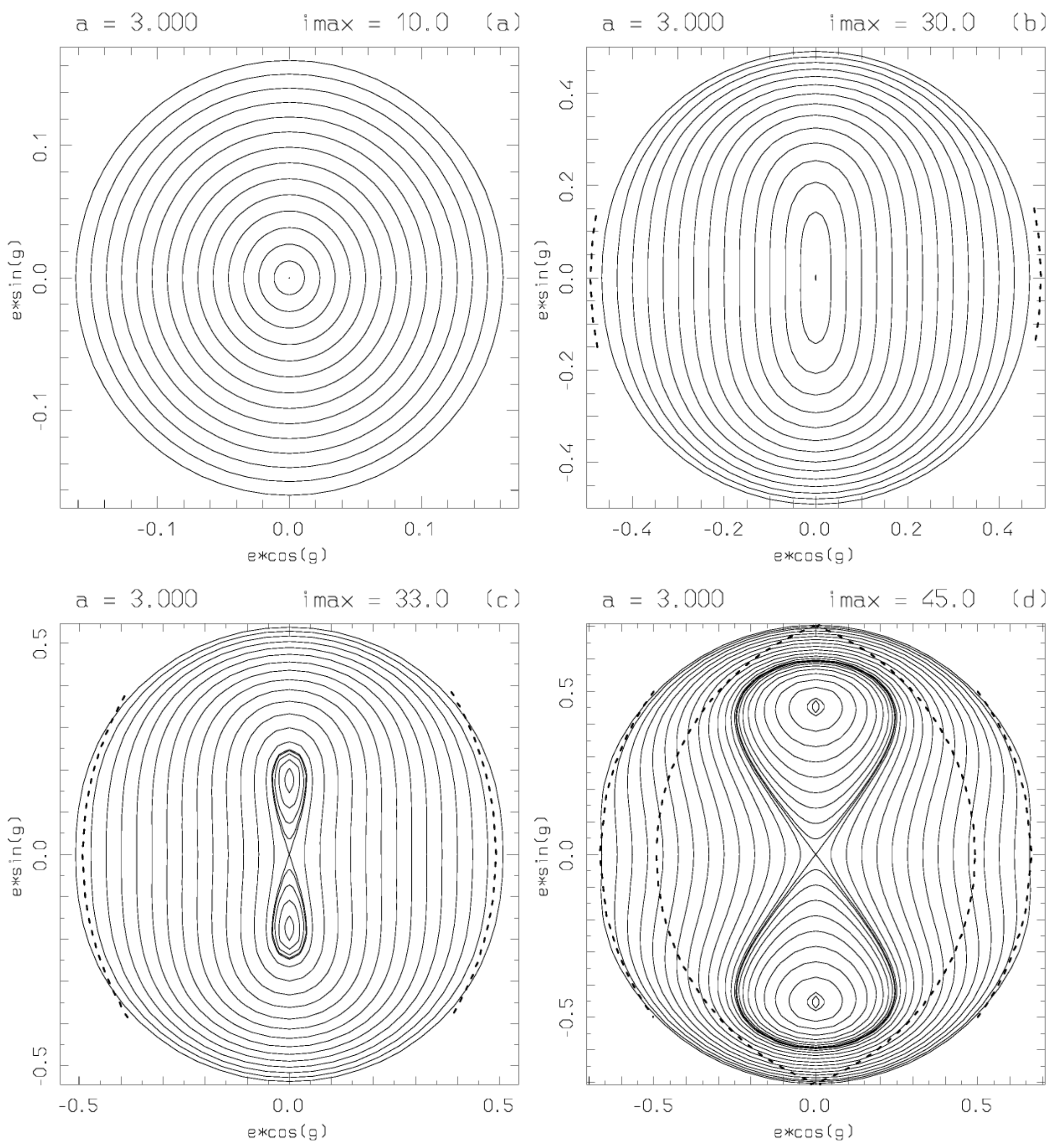
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial H} \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \Omega}$$

KL – Hamiltonian

- 1 degree-of-freedom: only depends on (*G*,*g*)
- Constant of motion: Delaunay H = H(*a*,*e*,*i*)
- Eccentricity & inclination coupled via H = const
- 2 extremal values:
  - H(*a*, *e*, *i*=0) → e_{max}
  - H(*a*, *e*=0, *i*) → i_{max}
- Analytical approximate Hamiltonian

$$K_0 = \sum_k \left(\frac{m_k}{16 a_k^3} \right) a^2 \Big[(2 + 3e^2)(3\cos^2(i) - 1) + 15 e^2 \sin^2(i) \cos(2\omega) \Big]$$

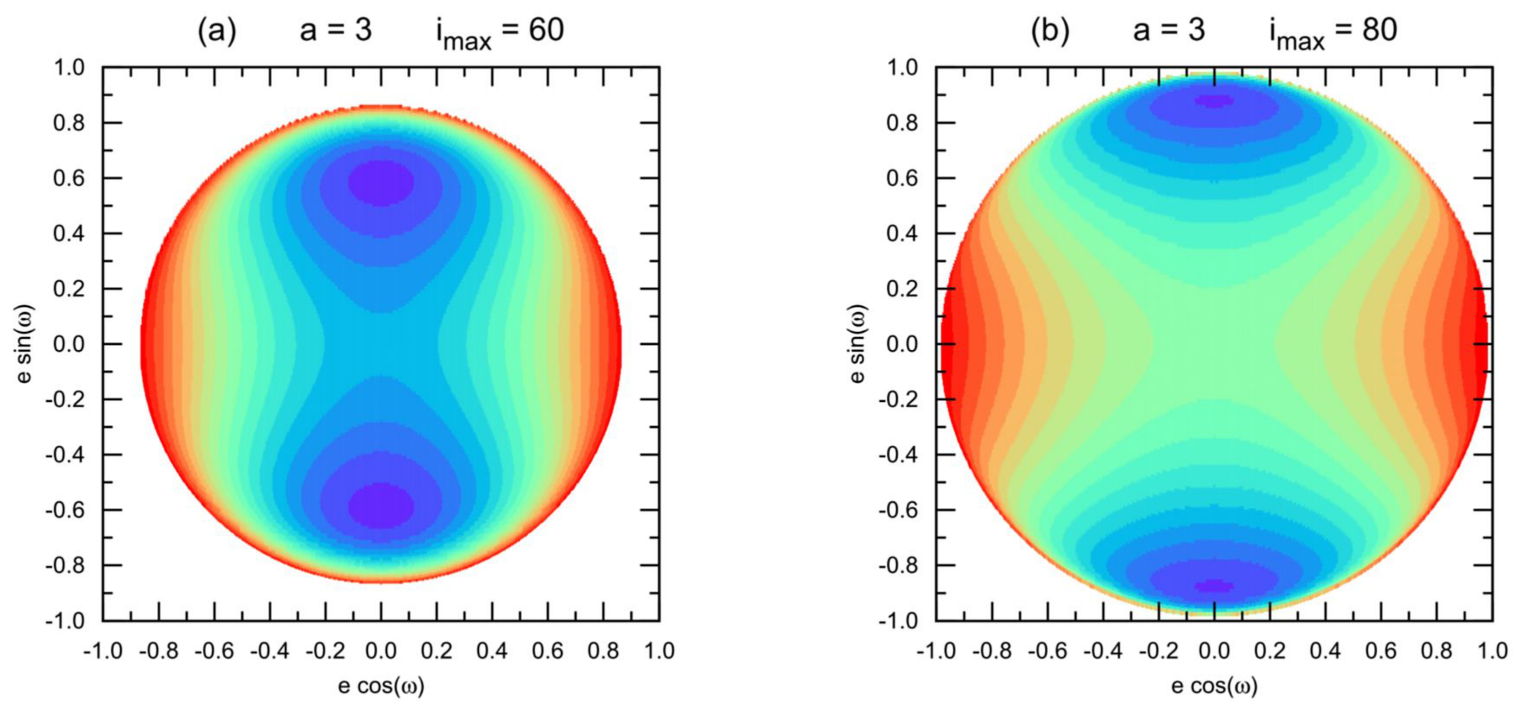
KL – phase space



Morbidelli (2002)

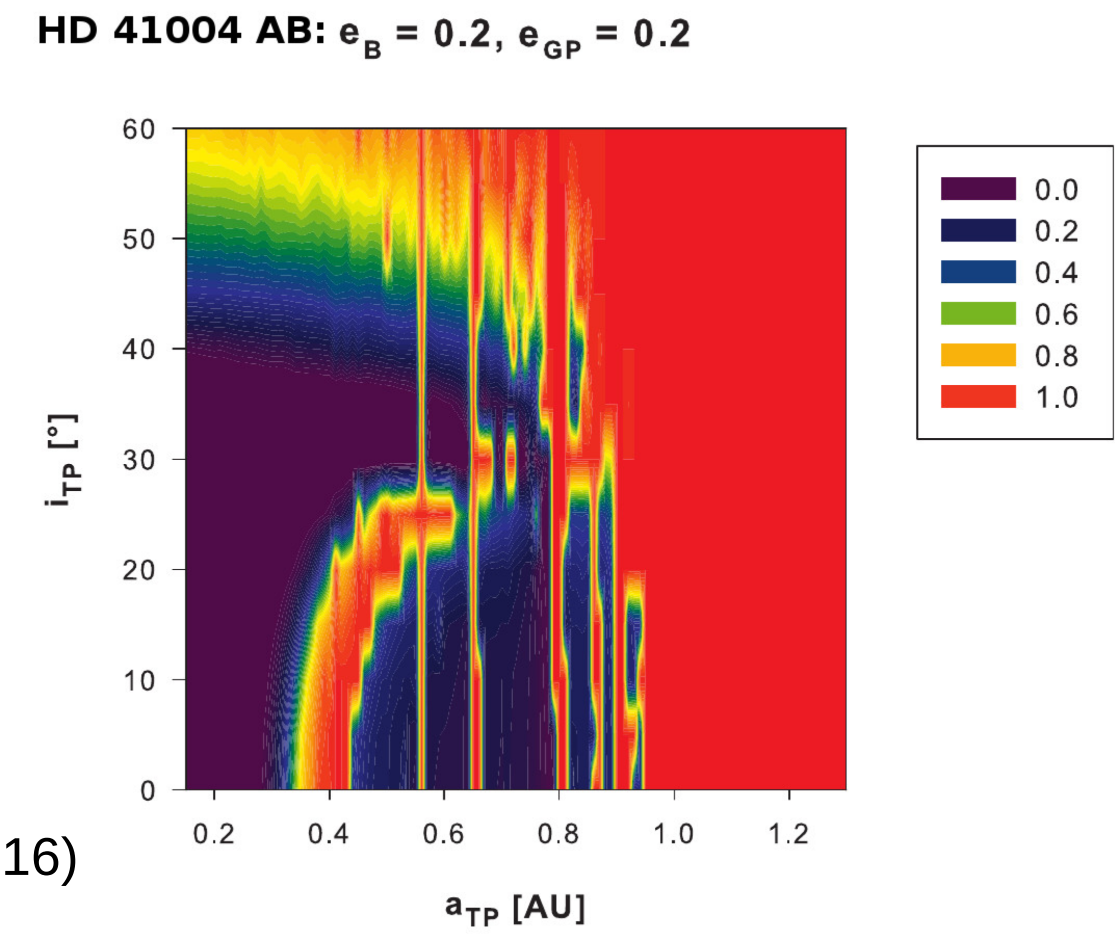
KL – dynamical mechanism

- At critical inclination $i_{crit} \approx 39.2°$ origin unstable fixed point
- (h,k) space: libration of ω about 90° / 270° fixed points
- Animation



KL – applications

- Artificial Earth satellites
- Irregular natural satellites
  (Brozovic,+ 2009, 2011, 2017)
- Exoplanets: Hot Jupiters, mis-aligned planets



Pilat-Lohinger,+ (2016)

6. Evection resonance (LLER)

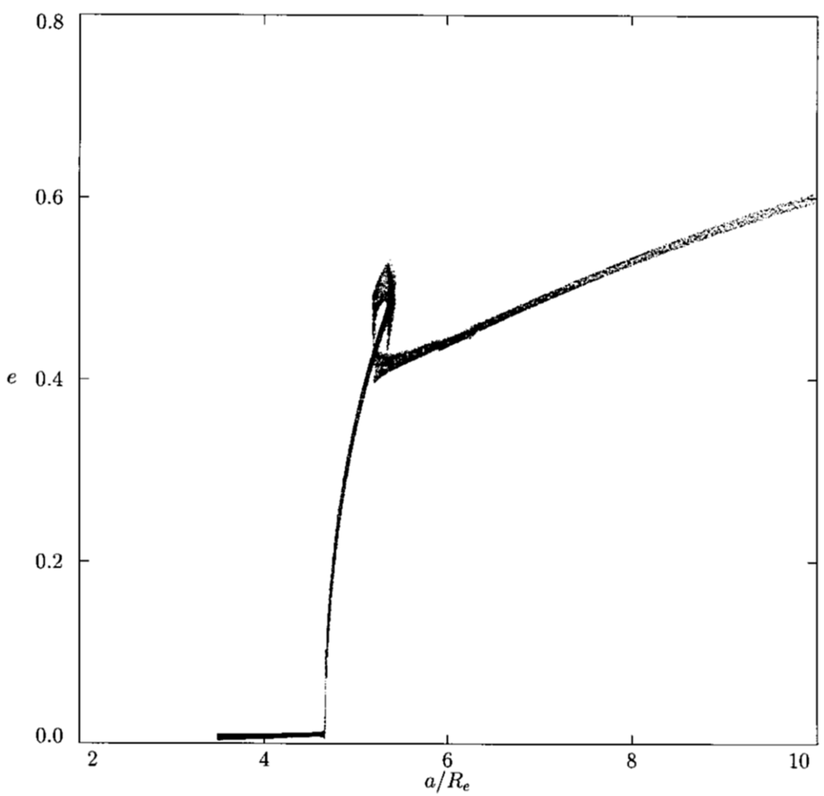
LLER – ingredients

- Multi-planet S-type binary star system
  - Host star m_A, secondary star m_B
  - 2 planets m_1, m_2 with $a_1 < a_2 \ll a_B$
- Star B on fixed Keplerian orbit
- Co-planar system

LLER – theoretical concepts

- Laplace-Lagrange evection resonance (LLER) (Touma & Sridhar 2015) → resonance between orbital precession frequency *g* and orbital frequency *n*
- Critical angle $\Phi_{res} = \tilde{\varpi} - n_B t$
- Connection to lunar evection resonance

Touma & Wisdom (1998)



LLER – Hamiltonian

- $H_{sec} = H_{LL} + H_{Bin} + H_{NL}$
- H_{LL} time independent
- H_{Bin} time-dependent periodic forcing
- H_{NL} non-linear higher order terms
- 3 LL modes (frequencies): ω_1, ω_2, $(\omega_1+\omega_2)/2$
- Coupling of n_B with modes

LLER – simulation

Without MMR

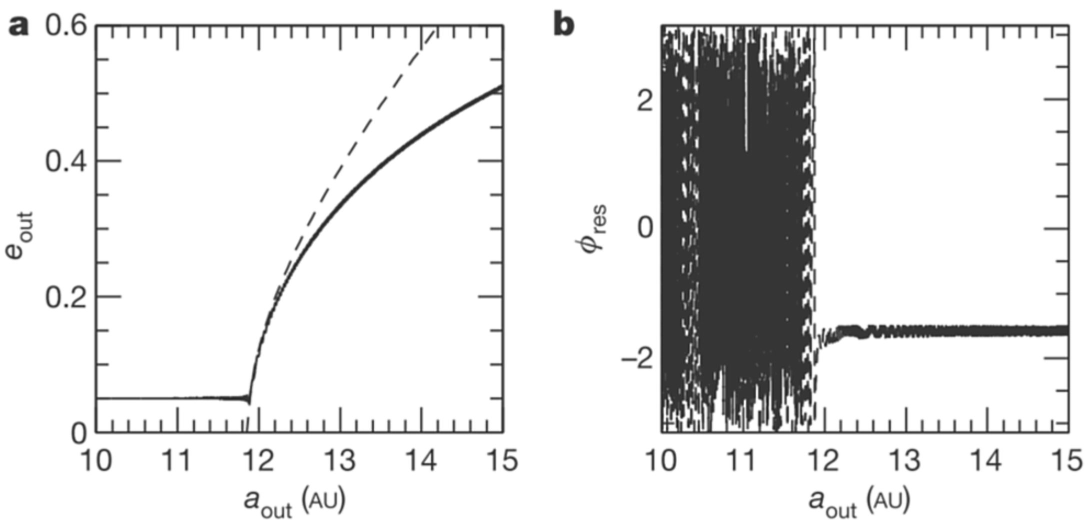
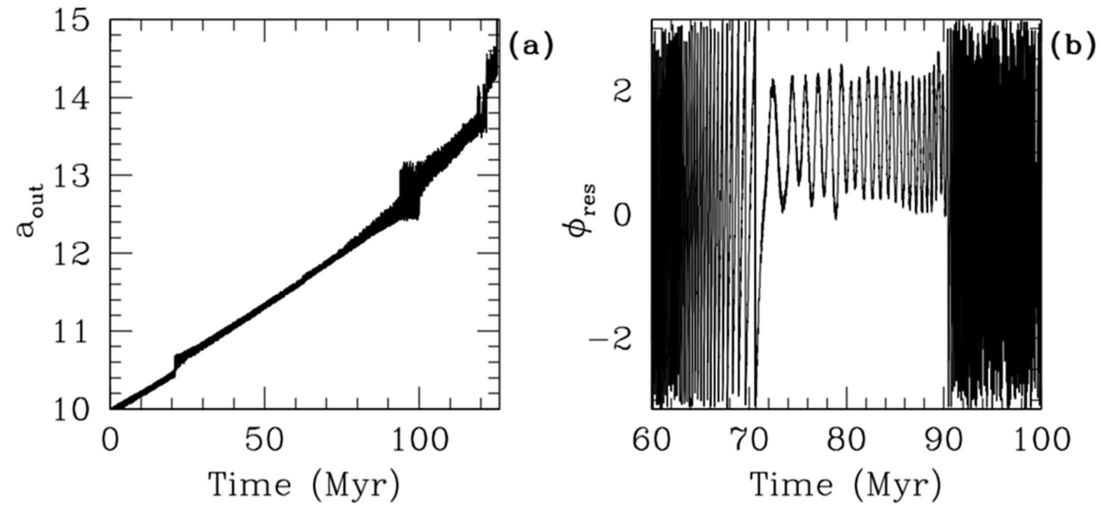


Figure 1 | Capture into the LLER. The fiducial *N*-wire experiment was performed with forced exponential migration[17], $a_{out}(t) = a^i_{out}\exp[t/\tau]$, with $a^i_{out} = 10$ AU and $\tau = 10^4 T_b$. **a**, Growth of e_{out} when it is captured in the migrating LLER. The dashed line is the prediction from the analytical fourth-order theory presented in Supplementary Information A. **b**, ϕ_{res} transitions from circulation to libration around 90° when captured in LLER.

With MMR



Touma & Sridhar (2015)

LLER – phase space

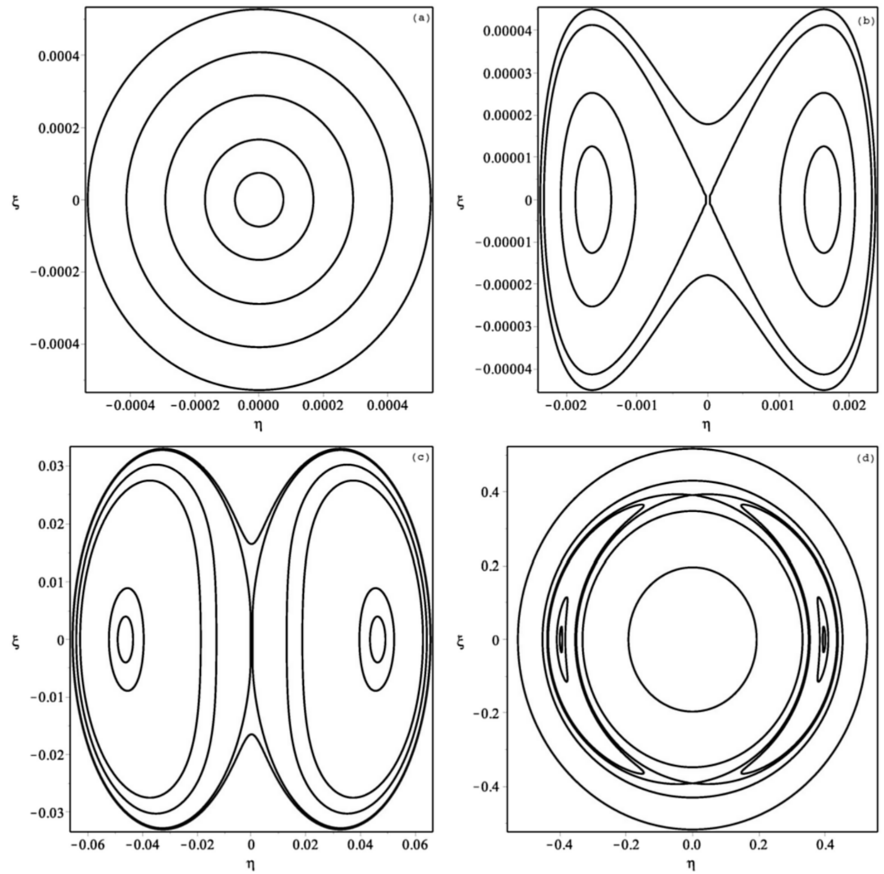
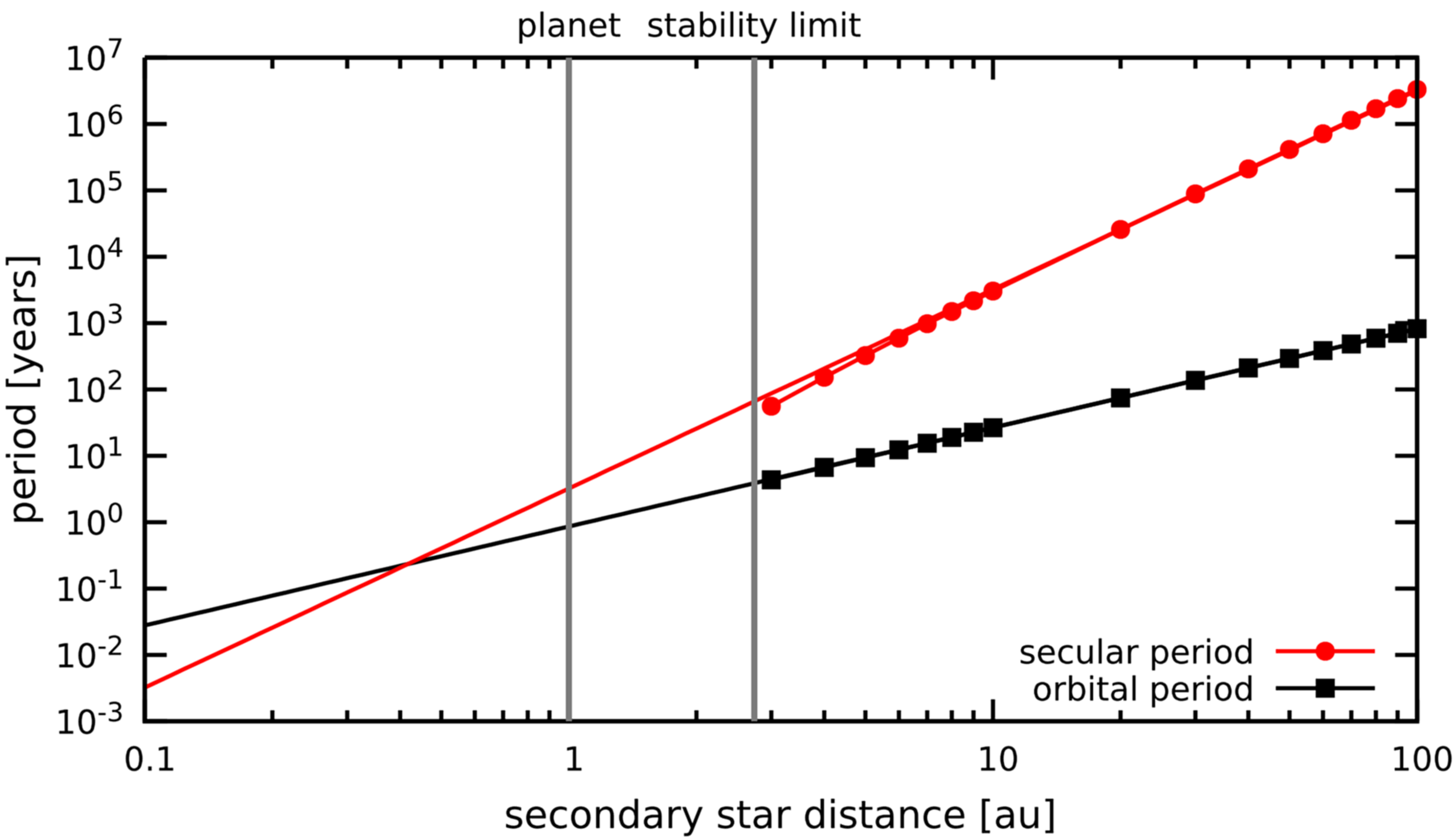


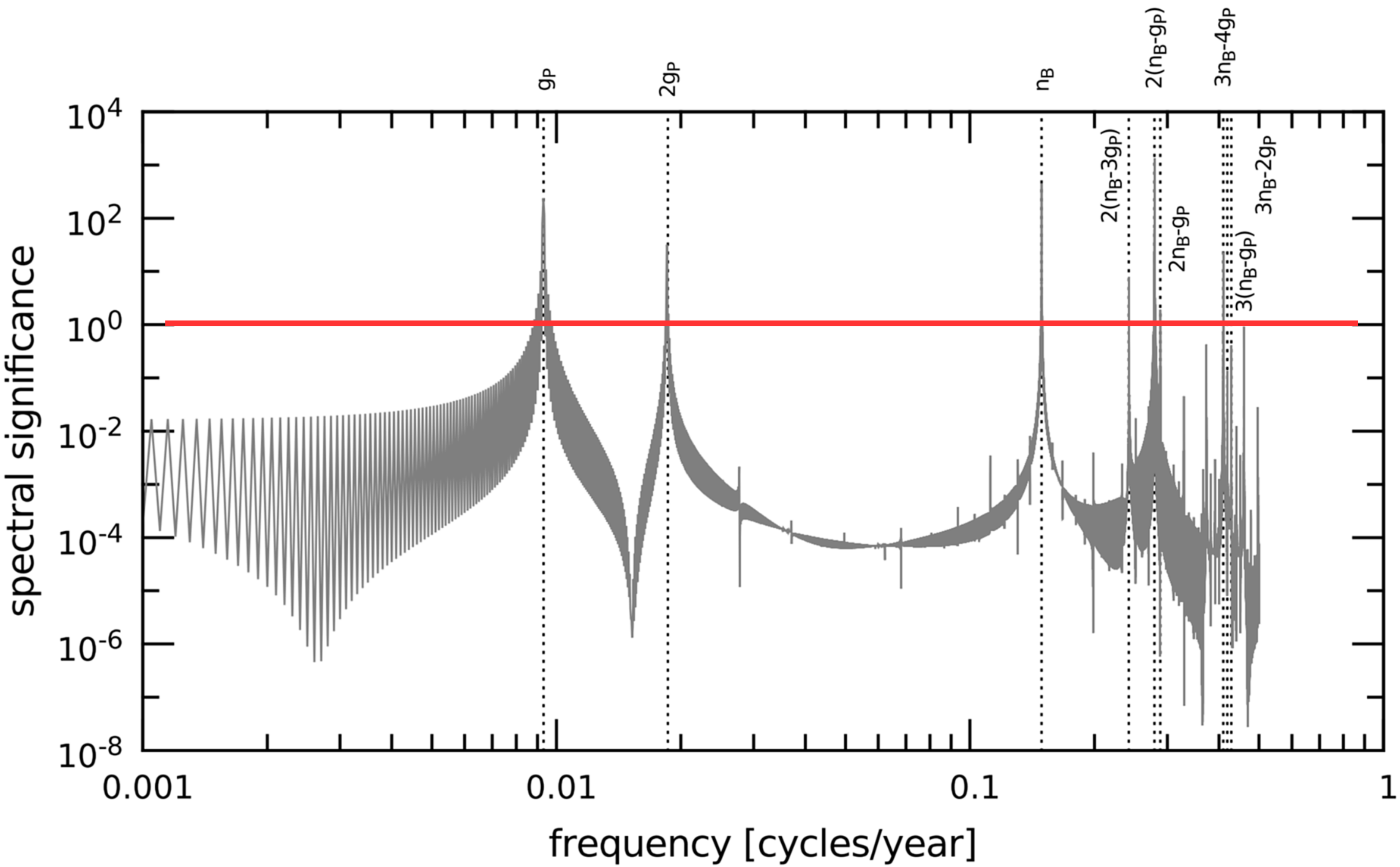
Figure S4: **Phase Space with Migrating Planet.** Isocontours of $H_{\rm nf}$ at different times showing bifurcations of equilibria and the emergence of islands where capture is probable. Note: both ξ and η have been rescaled by a factor $\sqrt{m_2\sqrt{GM_A a_2}}$, in order to turn them into eccentricity–like variables. **(a)** At $a_2 = 11$ AU the origin is stable with circulating orbits around it. **(b)** The origin goes unstable at $a_2 = 11.88$ AU and two LLER islands appear. **(c)** At $a_2 = 11.894$ AU the origin is about to go stable again. **(d)** At $a_2 = 13$ AU we are past the second bifurcation; there is an inner circulating zone surrounded by two libration lobes.

Touma & Sridhar (2015)

LLER – single planet

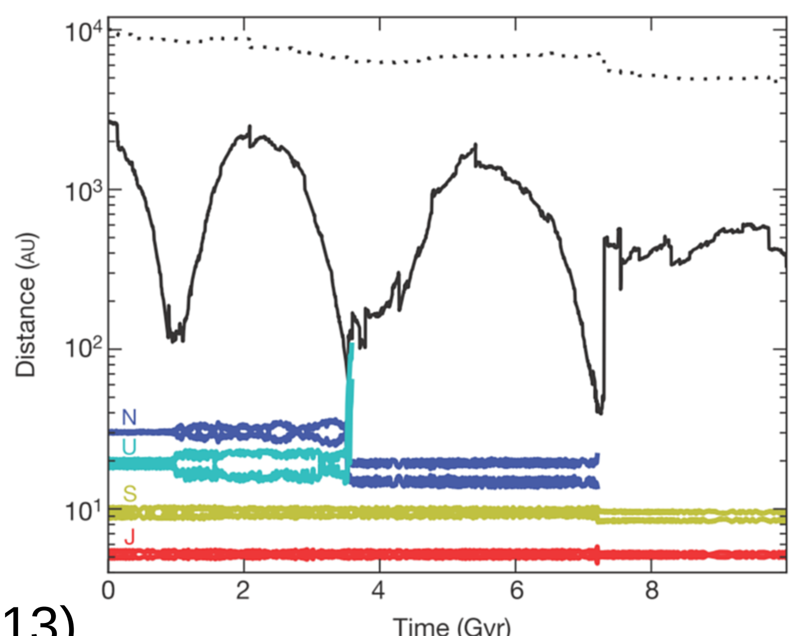


LLER – non-linear resonances

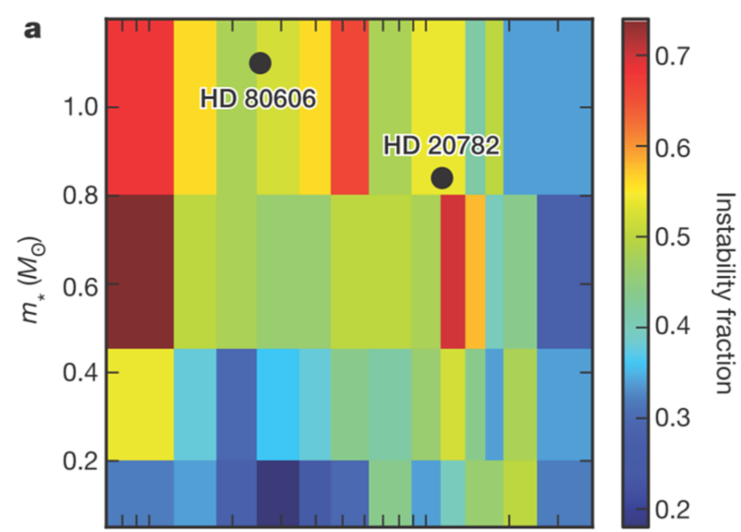


LLER – alternative mechanism

- No planetary migration
- Changing orbit of “wide” binary (a_B > 1000 AU) by galactic tide and passing stars



Kaib,+ (2013)



1000 < a_B [AU] < 30000

Summary

- **Kozai-Lidov resonance:**
  - Secular resonance between orbital precession frequencies
  - Periodic exchange between $e(t)$ & $i(t)$
- **Evection resonance:**
  - Secular resonance between orbital precession freq. and orbital freq. of a binary star
  - Important for multi-planet systems
- **Other resonances:**
  - Secondary resonances
  - …

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