The Effect of Resonances

Part 1

Topics overview

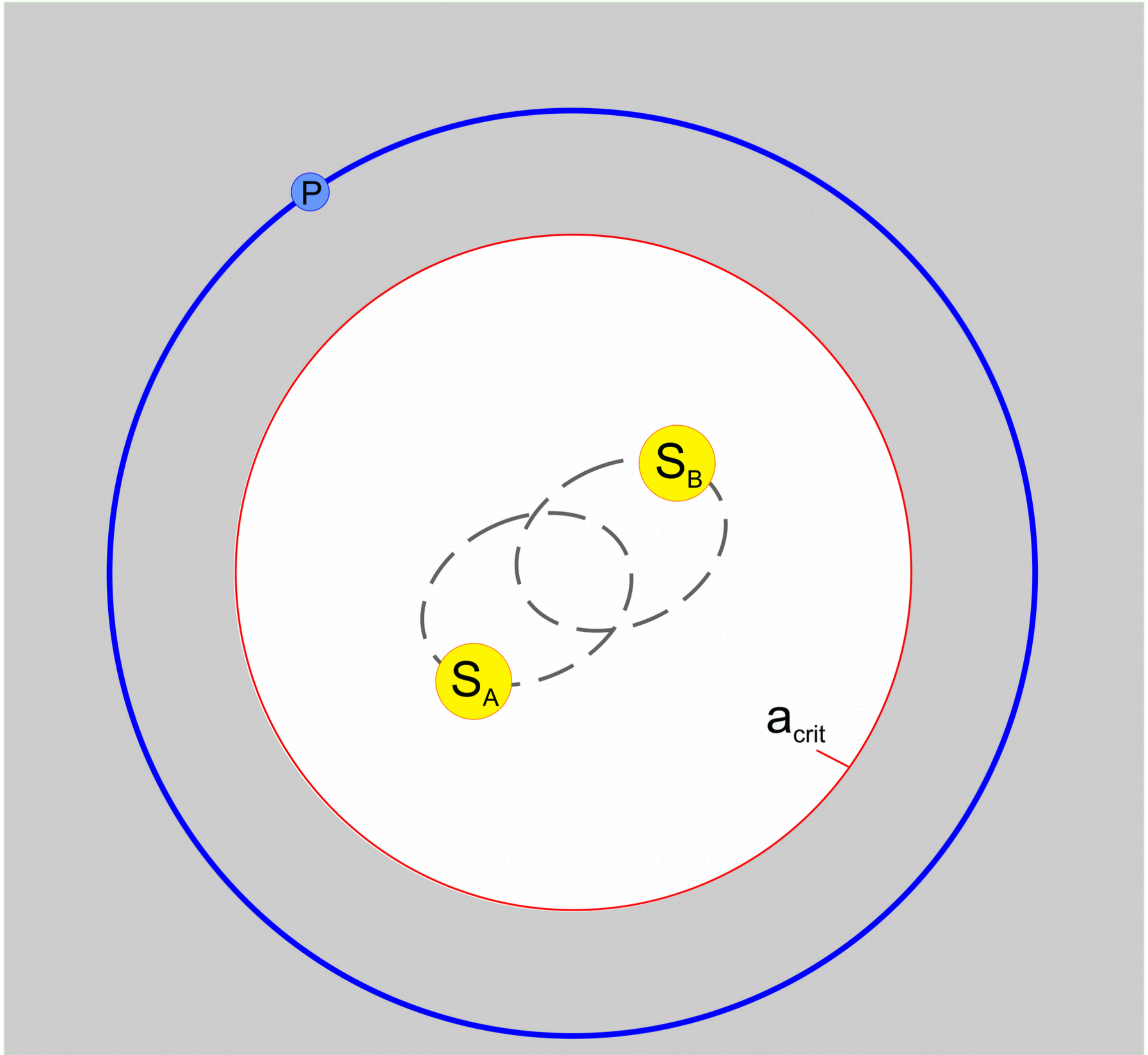
1. Definition and examples of “resonances”
2. Disturbing function
3. Mean-motion resonance (MMR)
4. Secular resonance (SR)
5. Kozai-Lidov resonance
6. Evection resonance
7. Other resonances

Recap

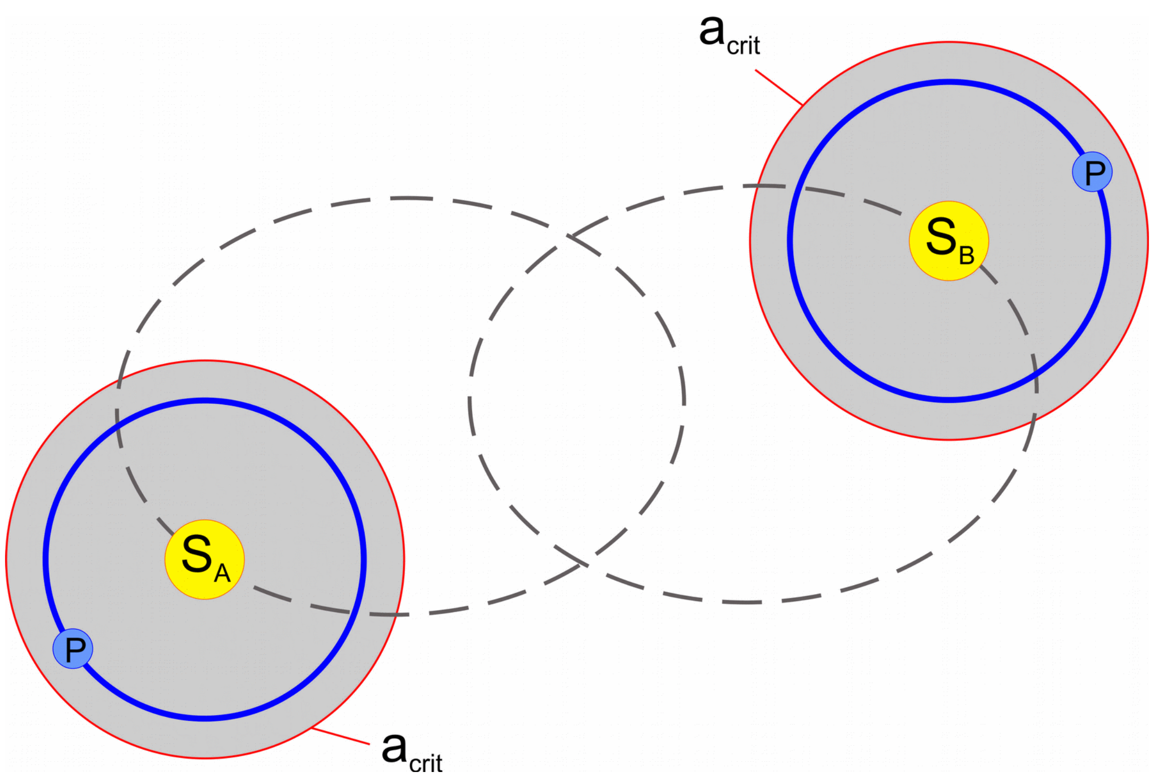
- Habitability → long-term stability → resonances
- Multi-stellar systems with exoplanets: 2, 3, 4 stars

  https://en.wikipedia.org/wiki/Star_system
  http://www.univie.ac.at/adg/schwarz/multiple.html
- Binary stars and hierarchical systems

Binary stars – domains of regular motion



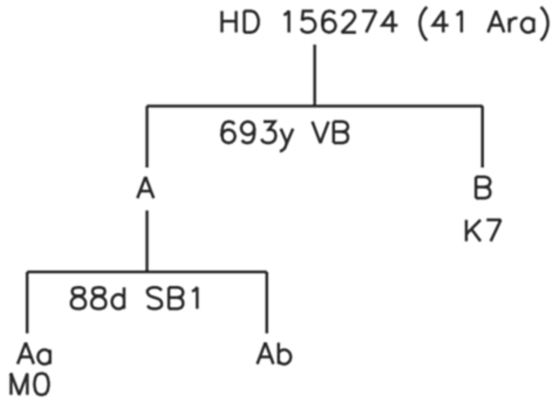
P-type system = circumbinary



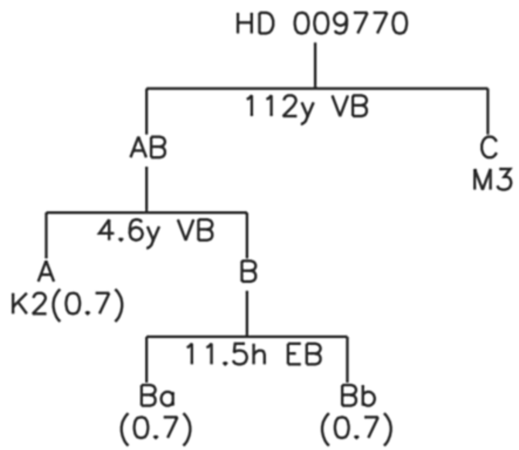
S-type system = circumstellar

Higher multiplicity stellar systems

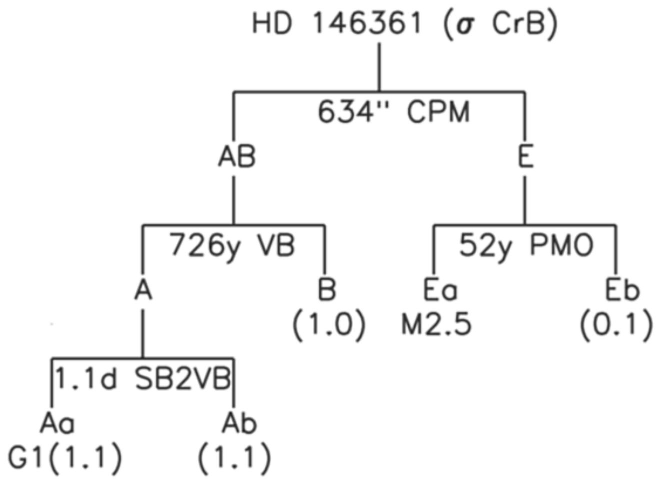
Triple system



Quadruple system

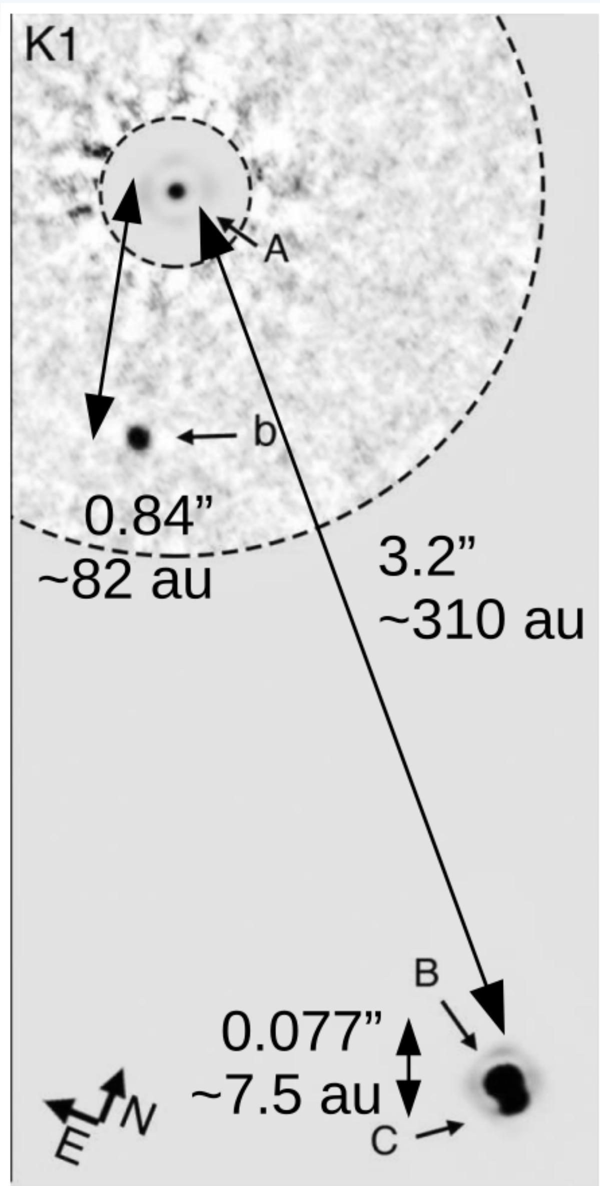


Quintuple system



Raghavan,+ (2010)

Higher multiplicity – HD 131399



- Direct imaging (NIR adaptive optics + coronograph)
- Dynamical configuration: Ab + BC
- Star A: A1V (1.82 M_0)
- Star B: G (0.96 M_0)
- Star C: K (0.6 M_0)

Wagner,+ (2016)

1. Resonances

Resonances

Definition

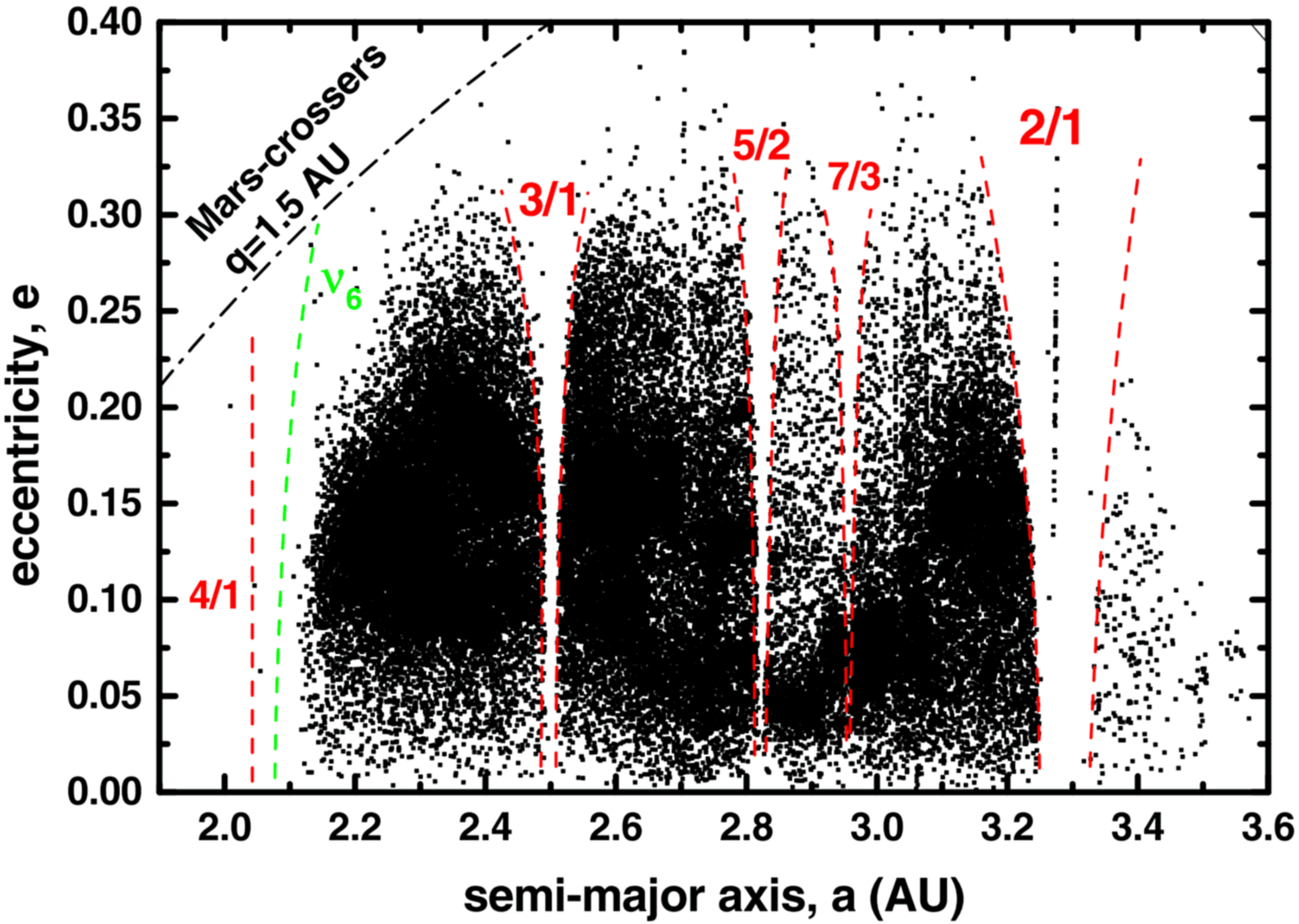
Two frequencies f_1 and f_2 are in resonance if their ratio can be expressed as a rational number

$$\left|\frac{f_1}{f_2}\right| = \frac{p}{q}, \qquad p, q \in \mathbb{N}$$

Types of resonances

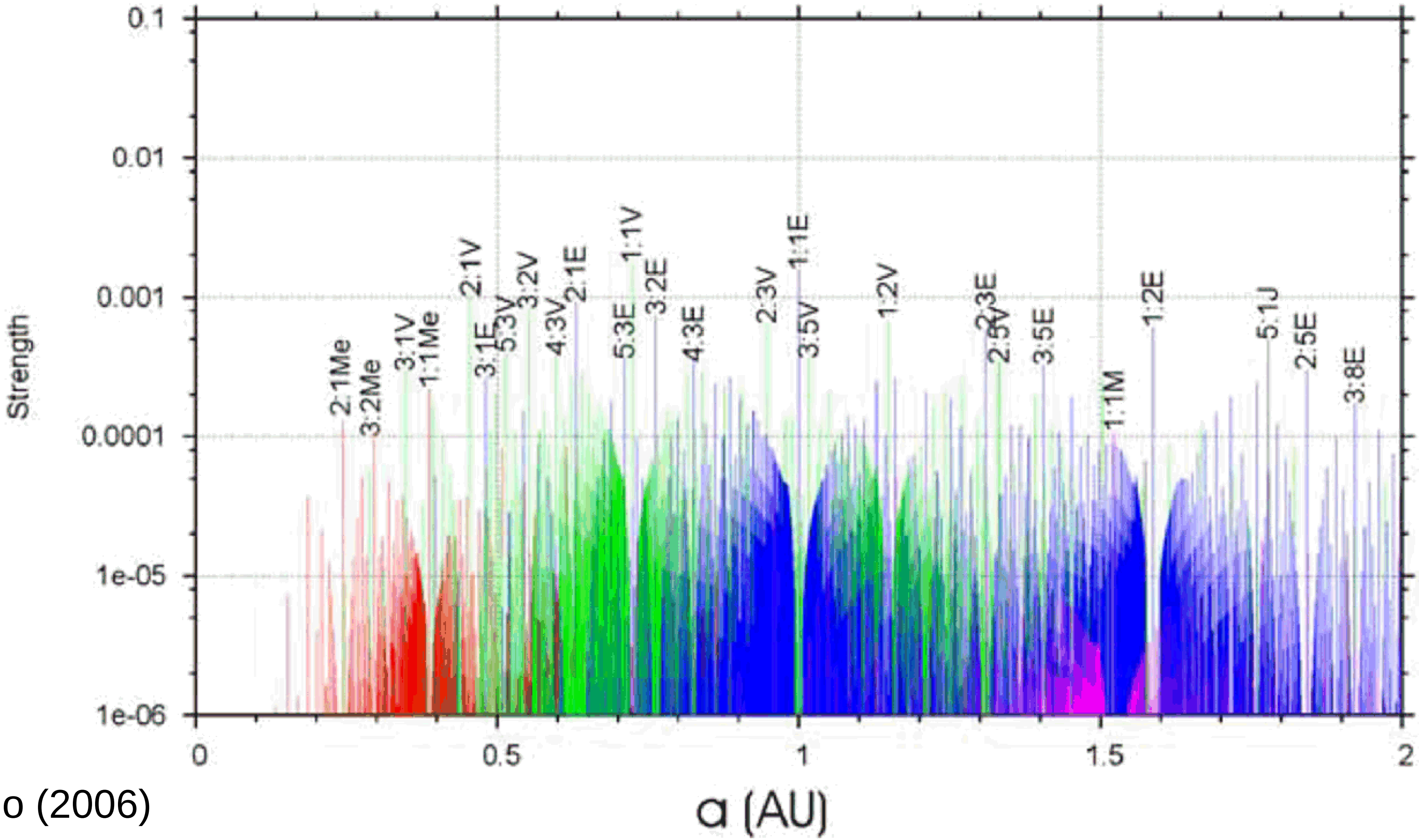
- **Mean-motion resonance** (MMR)
  - Orbital frequencies of two bodies
- **Three-body resonance**
  - Orbital frequencies of three bodies
- **Secular resonance** (SR)
  - Precession frequencies (perihelion, node) of ≥ 2 bodies
- **Spin-orbit resonance**
  - Orbital and spin frequency of same body
- **Gravitational resonance**
  - Orbital and spin frequency of different bodies

Examples – MMR



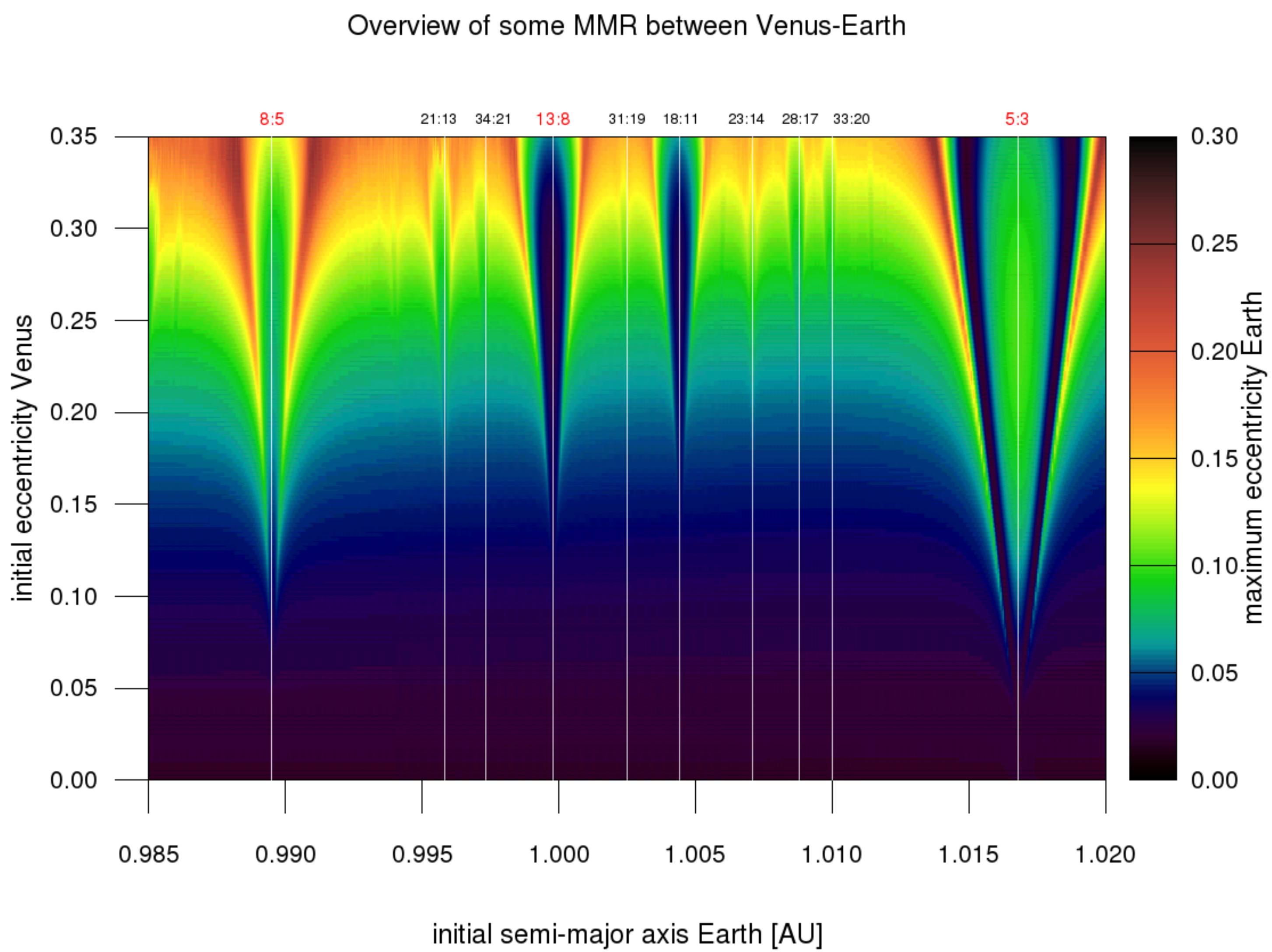
Tsiganis (2010)

Examples – MMR



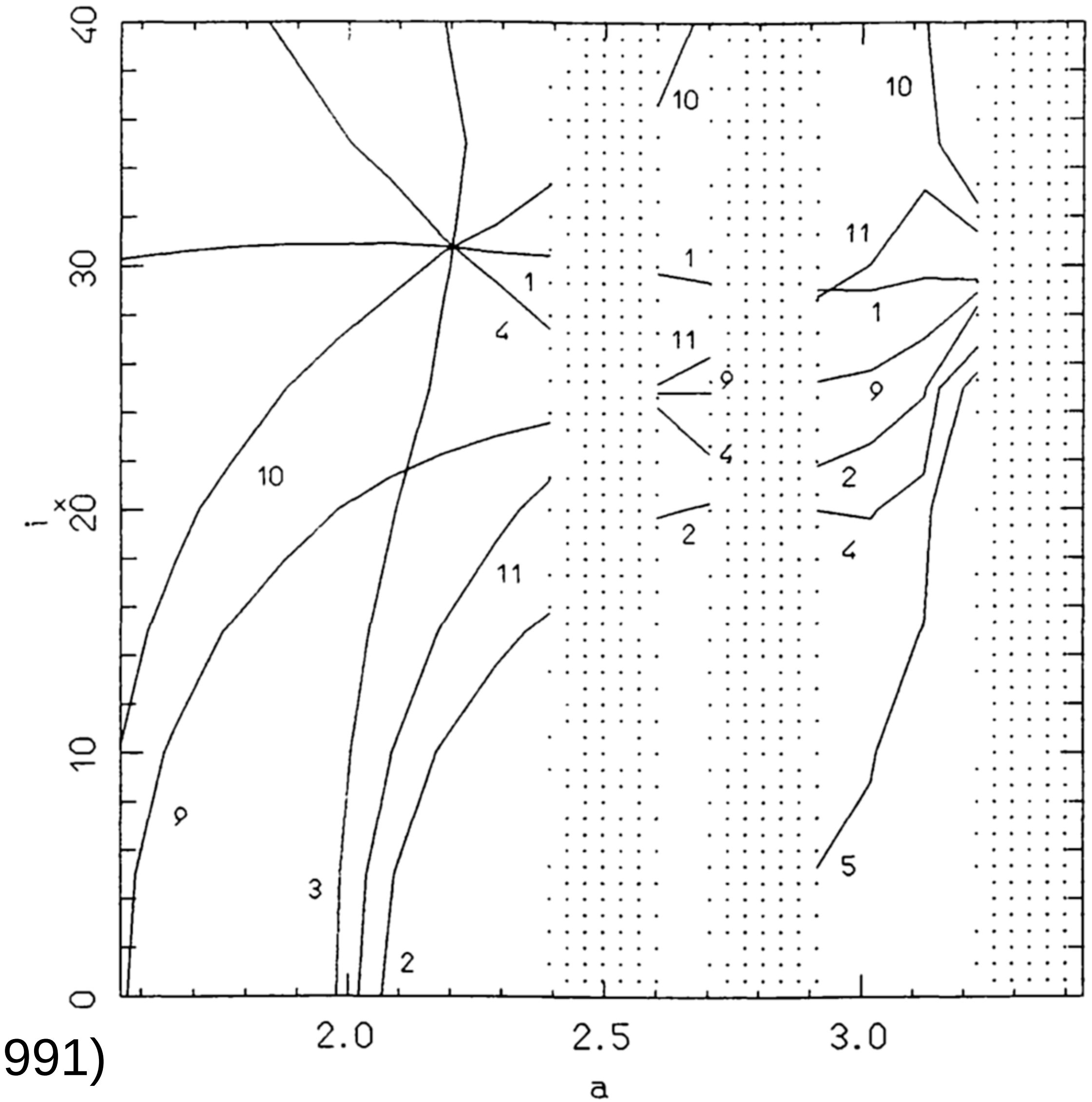
Gallardo (2006)

Examples – MMR



Bazsó,+ (2010)

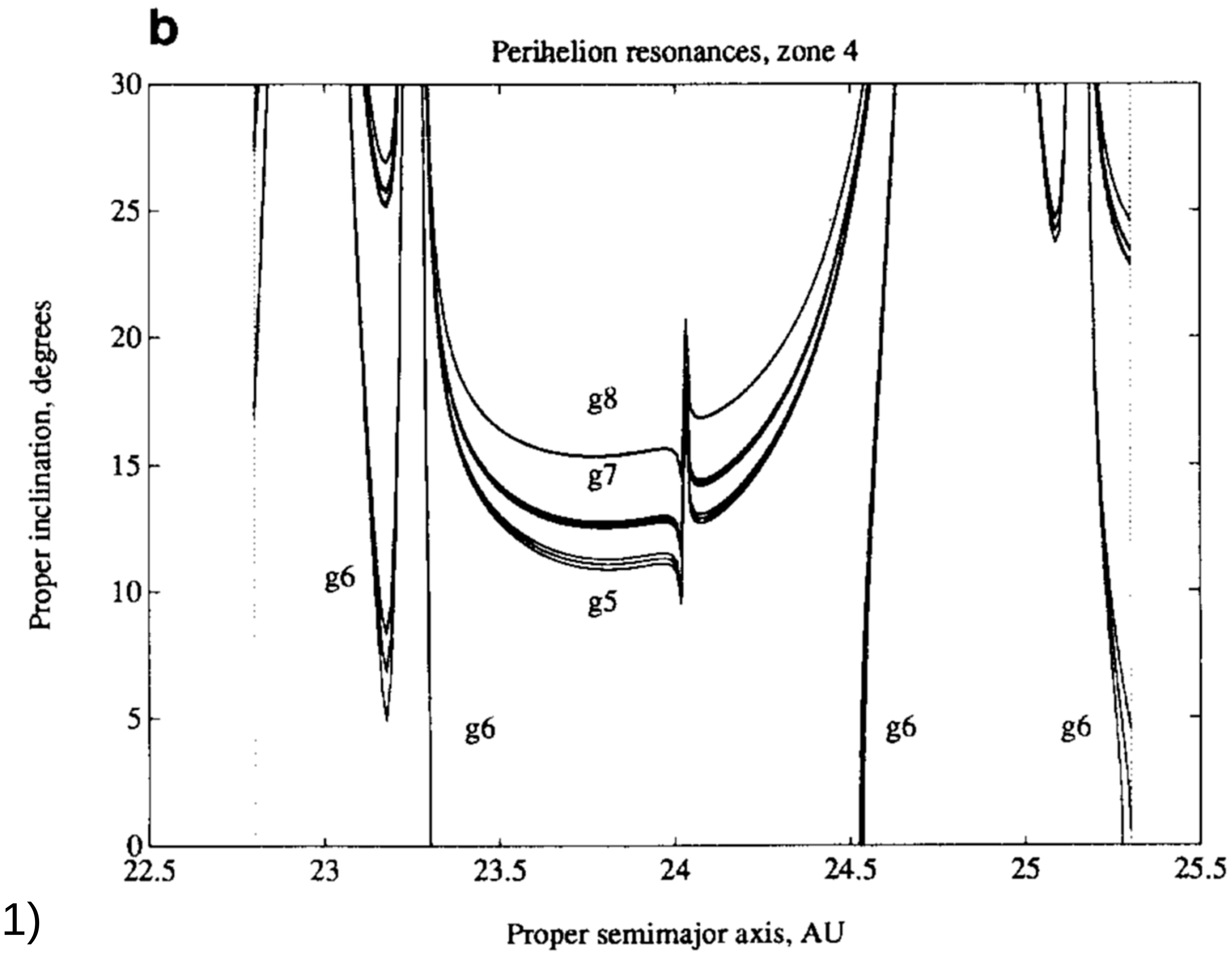
Examples – SR



Morbidelli & Henrard (1991)

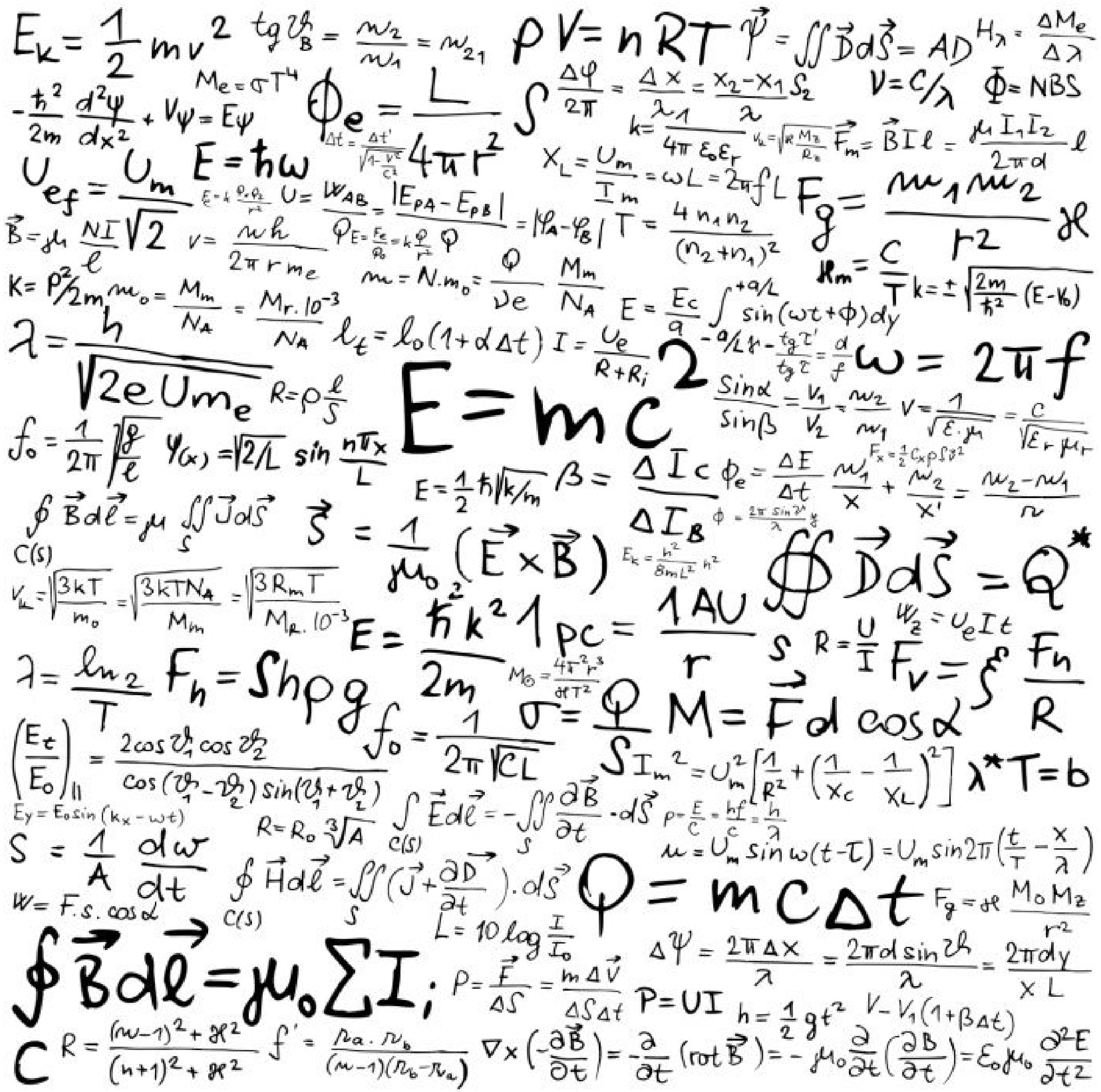
1: $g - g_5 = 0$
2: $g - g_6 = 0$
3: $s - s_6 = 0$
4: $g + s - g_5 - s_6 = 0$
5: $g + s - g_6 - s_6 = 0$
9: $2g - g_5 - g_6 = 0$
10: $g - s - g_5 + s_6 = 0$
11: $g - s - g_6 + s_6 = 0$

Examples – SR



Knezevic,+ (1991)

2. Disturbing function



Disturbing function – basics

- Basis for analytical approximations for $N \geq 3$ bodies
- Gradient of perturbing potential
- Equations of motion in heliocentric Cartesian coordinates

$$\ddot{\mathbf{r}}_i = -G(M + m_i)\frac{\mathbf{r}_i}{\|\mathbf{r}_i\|^3} + \sum_{n=1}^{N} G m_n \left(\frac{\mathbf{r}_n - \mathbf{r}_i}{\|\mathbf{r}_n - \mathbf{r}_i\|^3} - \frac{\mathbf{r}_n}{\|\mathbf{r}_n\|^3} \right)$$

Disturbing function – variables

- Express disturbing function in orbital elements
- Definition (Stiefel & Scheifele, 1971)
  an orbital element in the 2-body problem
  is a linear function of time: $f(t) = a + b\,t$
  - $(a, e, i, \omega, \Omega)$ = const.
  - $M = n\,(t - t_0) = M_0 + n\,t$ … only time dependent function
- Slow non-linear time-variation for $N>2$ body problem

Disturbing function – series expansions

Development in different coordinate systems:

- Heliocentric coord.
  - Murray & Dermott (1999), Ellis & Murray (2000)
- Mixed barycentric–heliocentric coord.
  - Laskar & Robutel (1995)
- Jacobi coord.
  - Mardling (2013)

Disturbing function – application

- Lagrange planetary equations
- Express time variation of orbital elements by:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \varepsilon}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{na^2 e}\left(1-\sqrt{1-e^2}\right)\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial \varpi}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\left(\frac{\partial \mathcal{R}}{\partial \varpi} + \frac{\partial \mathcal{R}}{\partial \epsilon}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathcal{R}}{\partial \Omega}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial i}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \mathcal{R}}{\partial i}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{2}{na}\frac{\partial \mathcal{R}}{\partial a} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial i}$$

3. Mean-motion resonance (MMR)

MMR – terms in disturbing function

Table B.1. Zeroth-order arguments: direct part.

ID	Cosine Argument	Term
4D0.2	$j\lambda' - j\lambda + \varpi' - \varpi$	$ee' f_{10} + e^3 e' f_{11} + ee'^3 f_{12}$ $+ee'(s^2 + s'^2) f_{13}$
4D0.3	$j\lambda' - j\lambda + \Omega' - \Omega$	$ss' f_{14} + ss'(e^2 + e'^2) f_{15}$ $+ss'(s^2 + s'^2) f_{16}$
4D0.4	$j\lambda' - j\lambda + 2\varpi' - 2\varpi$	$e^2 e'^2 f_{17}$
4D0.5	$j\lambda' - j\lambda + 2\varpi - 2\Omega$	$e^2 s^2 f_{18}$
4D0.6	$j\lambda' - j\lambda + \varpi' + \varpi - 2\Omega$	$ee' s^2 f_{19}$
4D0.7	$j\lambda' - j\lambda + 2\varpi' - 2\Omega$	$e'^2 s^2 f_{20}$
4D0.8	$j\lambda' - j\lambda + 2\varpi - \Omega' - \Omega$	$e^2 ss' f_{21}$
4D0.9	$j\lambda' - j\lambda + \varpi' - \varpi - \Omega' + \Omega$	$ee' ss' f_{22}$
4D0.10	$j\lambda' - j\lambda + \varpi' - \varpi + \Omega' - \Omega$	$ee' ss' f_{23}$
4D0.11	$j\lambda' - j\lambda + \varpi' + \varpi - \Omega' - \Omega$	$ee' ss' f_{24}$
4D0.12	$j\lambda' - j\lambda + 2\varpi' - \Omega' - \Omega$	$e'^2 ss' f_{25}$
4D0.13	$j\lambda' - j\lambda + 2\varpi - 2\Omega'$	$e^2 s'^2 f_{18}$

Murray & Dermott (1999)

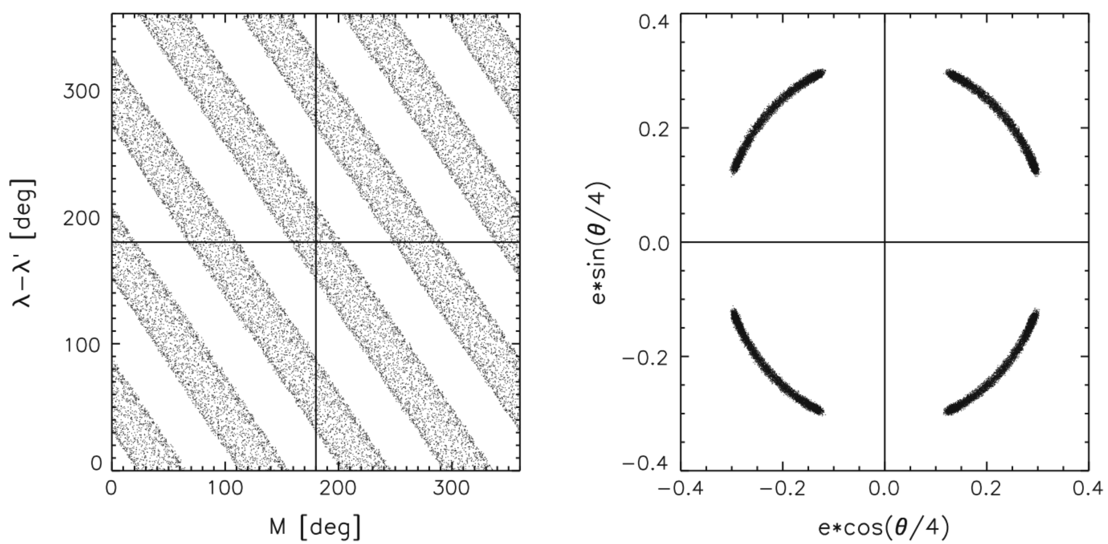
MMR – theoretical concepts

- MMR → orbital frequencies
- Critical angle of MMR
- Small divisor for $j_1 n + j_2 n' \approx 0$
- Resonance location a_{res} from 3rd Kepler law

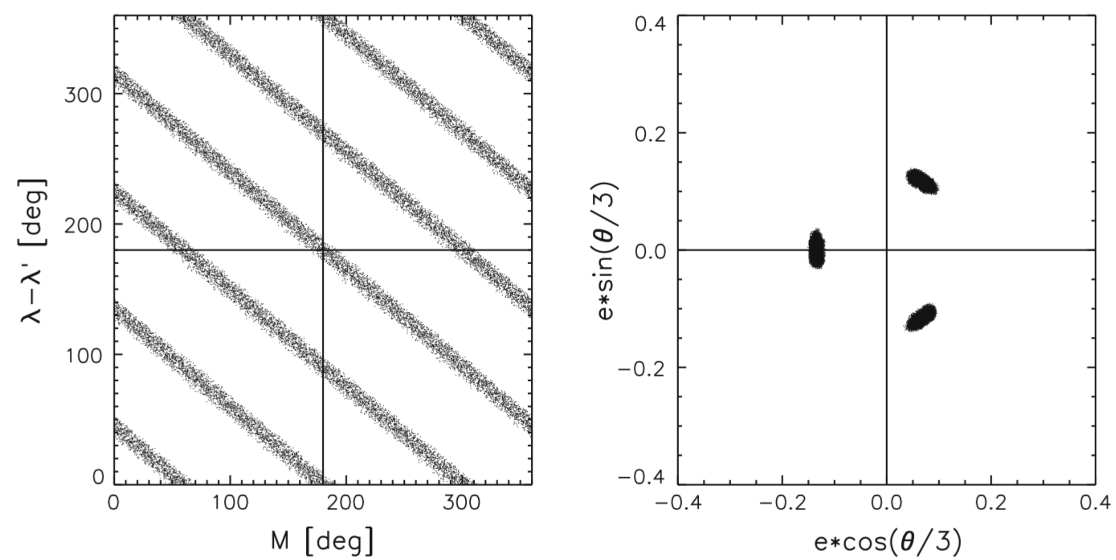
$$a_{\rm res} = a' \left(\frac{n'}{n}\right)^{2/3} \left(\frac{M+m}{M+m'}\right)^{1/3}$$

MMR – visualization

7:3 MMR



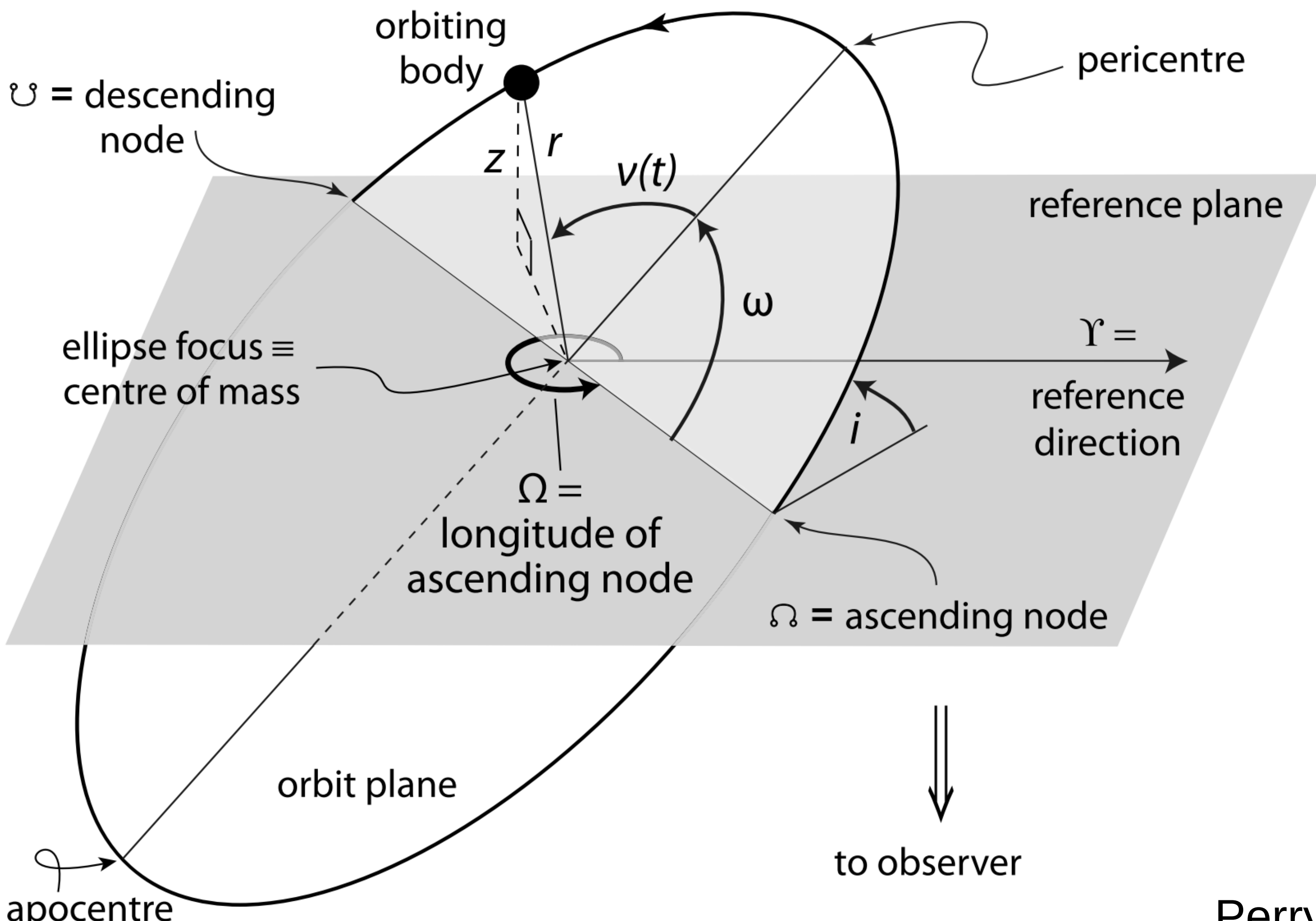
7:4 MMR



Érdi,+ (2012)

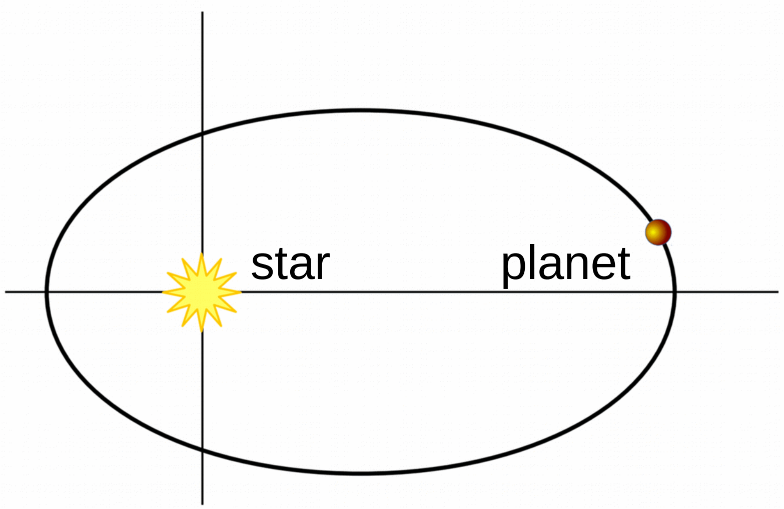
4. Secular resonance (SR)

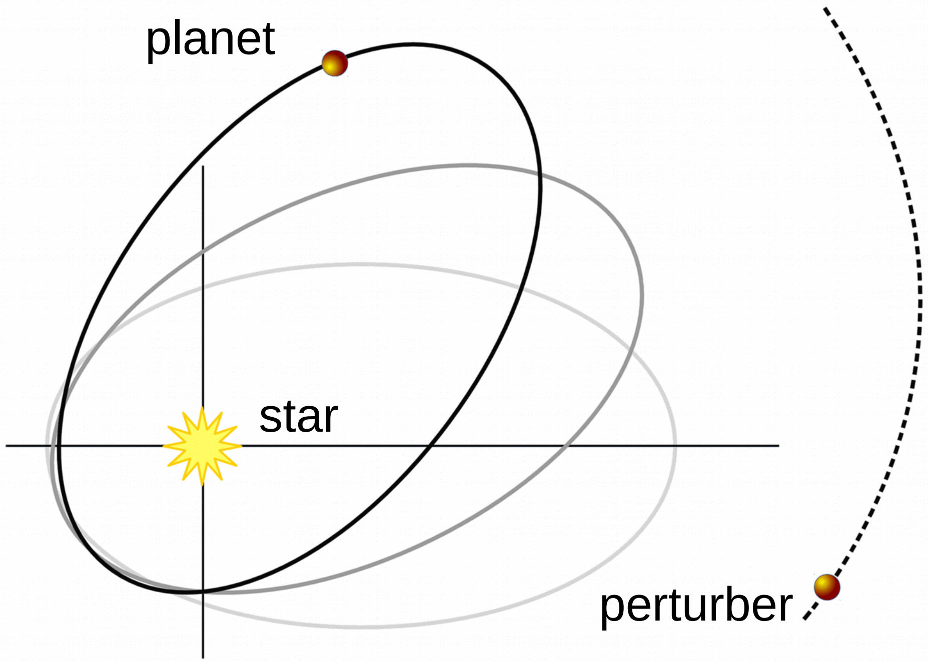
Secular resonances



Perryman (2011)

Secular resonances





SR – theoretical concepts

- SR → orbital precession frequencies
- Line of apsides, line of nodes
- Time-scale $T_{\mathrm{sec}} >> T_{\mathrm{rev}}$

SR – disturbing function

- Averaging principle
- Remove short period contributions
- Eliminate “fast” frequencies λ ~ *M*
- Averaged (secular) disturbing function

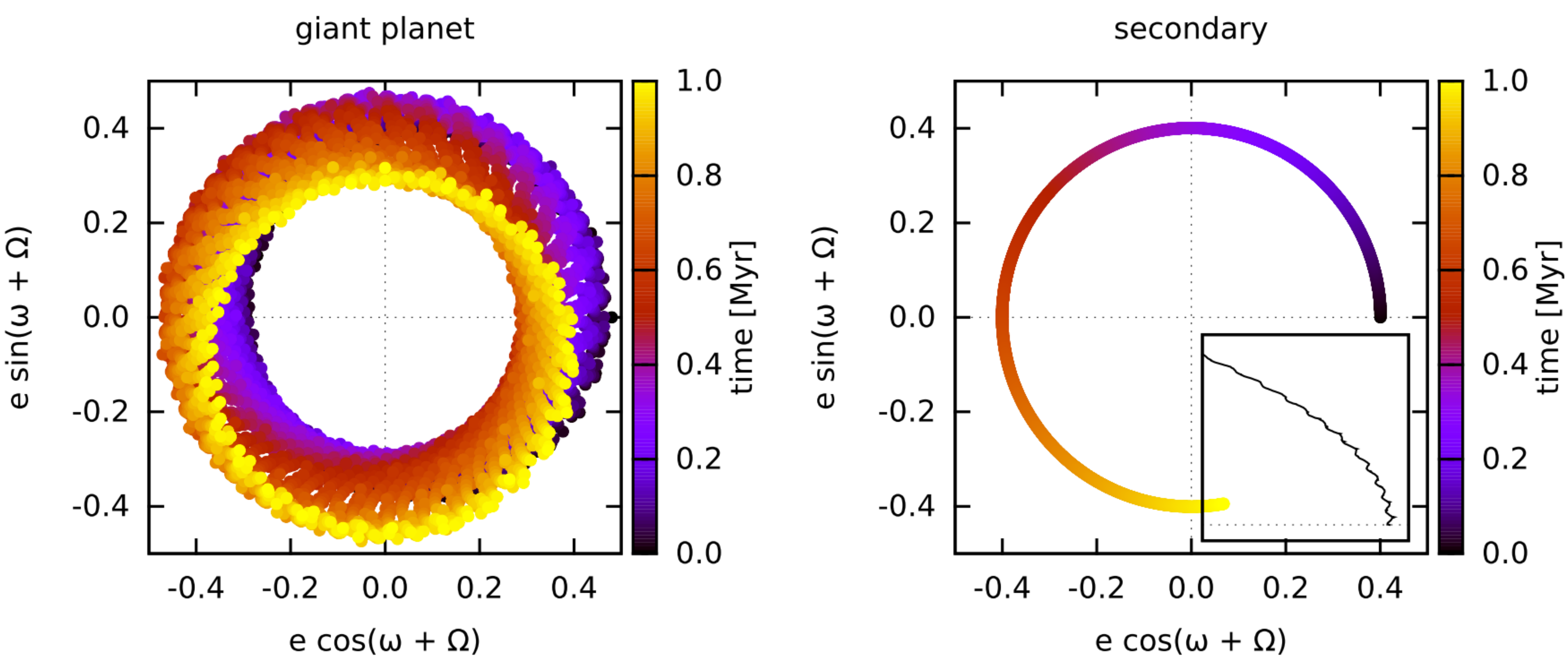
$$\mathcal{R}(a, e, i, \omega, \Omega, \lambda) \longmapsto \langle \mathcal{R} \rangle(-, e, i, \omega, \Omega, -) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(a, e, i, \omega, \Omega, \lambda) \, \mathrm{d}\lambda$$

SR – secular variables

- Laplace-Lagrange variables
- Decoupling of eccentricity / inclination (to lowest order)

$$h = e\sin(\omega + \Omega) \qquad p = \sin(i/2)\sin\Omega$$
$$k = e\cos(\omega + \Omega) \qquad q = \sin(i/2)\cos\Omega$$

SR – orbital precession



SR – solutions

- Equations of motion in variables (h,k) – system of linear differential equations
- Secular eigenfrequencies (eigenvalues) g_i
- Laplace coefficients $b_n^{(k)}(\alpha)$

$$\dot{\mathbf{h}} = \mathbf{A}\mathbf{k}$$

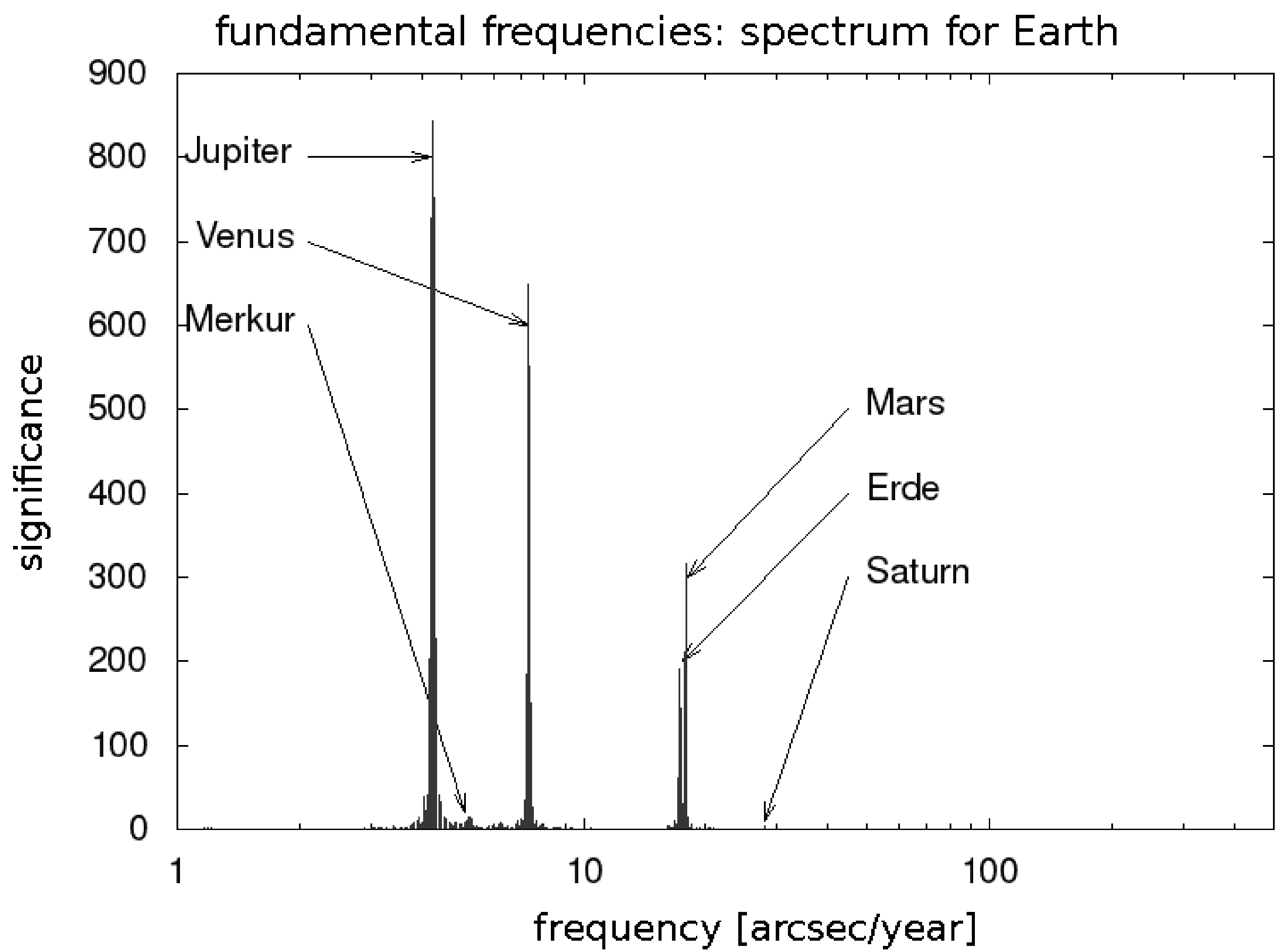
$$\dot{\mathbf{k}} = -\mathbf{A}\mathbf{h}$$

$$A_{j,j} = +\frac{1}{4} n_j \sum_{k=1}^{N} \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(1)}(\alpha_{j,k})$$

$$A_{j,k} = -\frac{1}{4} n_j \frac{m_k}{M + m_j} \alpha_{j,k}^2 \, b_{3/2}^{(2)}(\alpha_{j,k})$$

$$\det(\mathbf{A} - g\mathbf{1}) = 0$$

SR – frequencies



SR – test particle

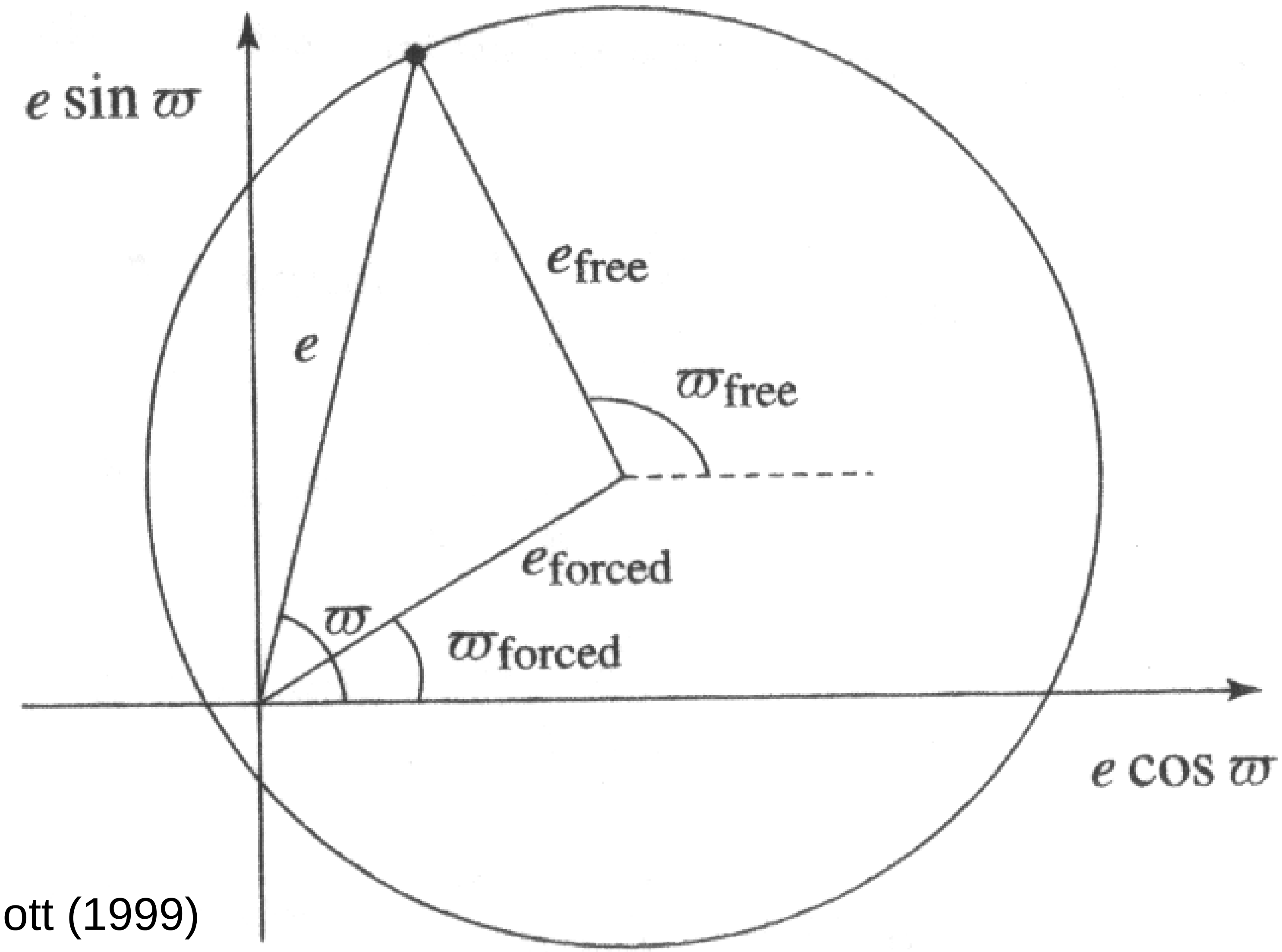
- Disturbing function for a TP with *N* massive perturbers
- Proper frequency *g* of TP
- General solution for TP in (*h*,*k*) variables
- Small divisor for $g - g_i \approx 0$
- Proper (free) + forced eccentricity / inclination

$$\mathcal{R} = n\,a^2 \left[\frac{1}{2} g \left(h^2 + k^2 \right) + \sum_{j=1}^{N} A_j \left(h\,h_j + k\,k_j \right) \right]$$

$$g = \frac{1}{4} n \sum_{j=1}^{N} \frac{m_j}{M}\, \alpha_j^2\, b_{3/2}^{(1)}(\alpha_j)$$

$$\begin{aligned} h(t) &= e_{\rm free} \sin(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \sin(g_i t + \varphi_i) \\ &= h_{\rm free}(t) + h_0(t) \\ k(t) &= e_{\rm free} \cos(gt + \varphi) - \sum_{i=1}^{N} \frac{\nu_i}{g - g_i} \cos(g_i t + \varphi_i) \\ &= k_{\rm free}(t) + k_0(t) \\ e_{\rm forced} &= \sqrt{h_0^2 + k_0^2} \\ i_{\rm forced} &= \sqrt{p_0^2 + q_0^2} \end{aligned}$$

SR – free / forced eccentricity



Murray & Dermott (1999)

Summary

- **Concepts:**
  - What is a resonance?
  - Which kinds of resonances exist?
- **Disturbing function:**
  - What is it representing?
  - Why is it useful?
- **Mean-motion resonances:**
  - What causes MMR?
- **Secular resonances:**
  - What causes SR?

References

- Bazsó, Dvorak, Pilat-Lohinger, Eybl, Lhotka (2010), CMDA 107, 63–76
- Ellis, Murray (2000), Icarus 147, 129–144
- Érdi, Rajnai, Sándor, Forgács-Dajka (2012), CMDA 113, 95–112
- Gallardo (2006), Icarus 184, 29–38
- Knezevic, Milani, Farinella, Ch. Froschle, Cl. Froeschle (1991), Icarus 93, 316–330
- Laskar, Robutel (1995), CMDA 62, 193–217
- Mardling (2013), MNRAS 435, 2187–2226
- Morbidelli, Henrard (1991), CMDA 51, 131–167
- Murray, Dermott (1999), Solar System Dynamics, Cambridge Univ. Press
- Perryman (2011), The Exoplanet Handbook, Cambridge Univ. Press
- Raghavan, McAlister, Henry, ++ (2010), ApJS 190, 1–42
- Stiefel, Scheifele (1971), Linear and Regular Celestial Mechanics, Spinger
- Szebehely (1984), CMDA 34, 49–64
- Tsiganis (2010), Eur. Phys. J. Spec. Topics 186, 67–89
- Wagner, Apai, Kasper, ++ (2016), Science 353, 673–678