Chaos Indicators

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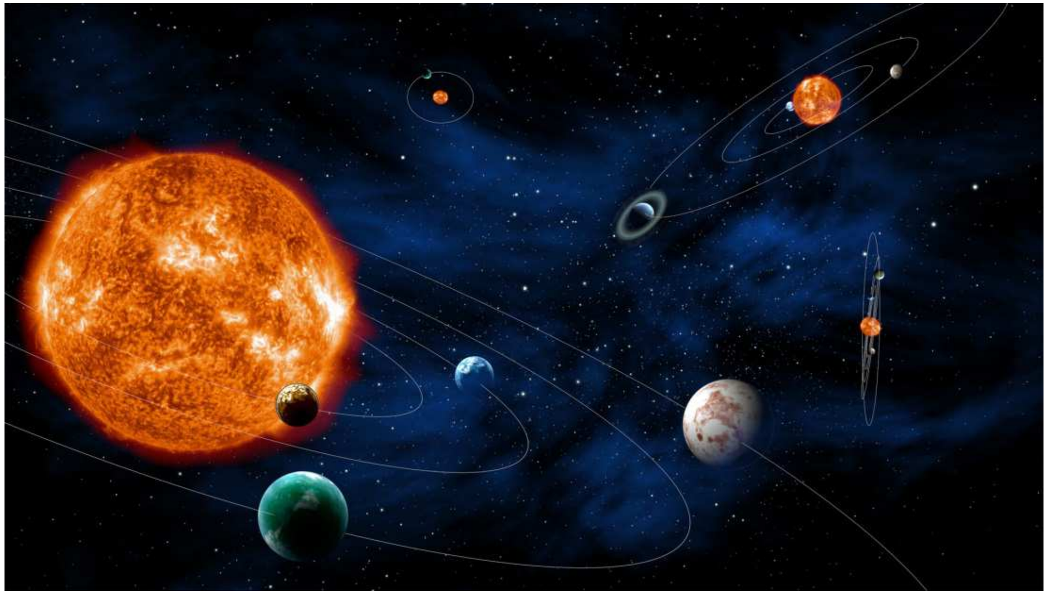
Introduction

- Major question in celestial mechanics:

  -> Stability of (multi) planetary systems

  - Stability issues analysed by
    - Laplace, Lagrange,
    - Gauss, Poincaré,
    - Kolmogorov, Arnold, Moser, …

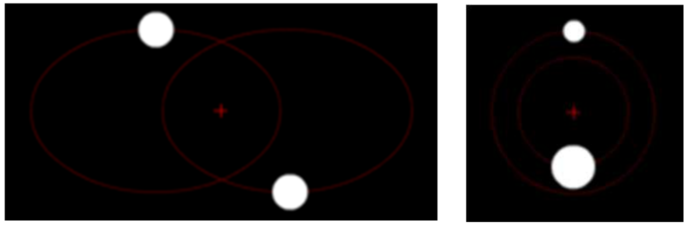


Credit: http://www.cosmos.esa.int/web/plato

- Multi-planetary systems:
  - generally N-body problems
- N-Body problem:
  - No general, analytical solution -> approximations -> perturbation theory

Introduction

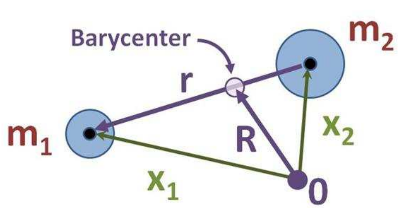
- 2-Body problem:
  - exactly solvable
  - Kepler's laws, Kepler equation, Kepler orbits
  - completely solved by Johann Bernoulli (1734)
  - elliptic, parabolic, hyperbolic solutions (conic sections)



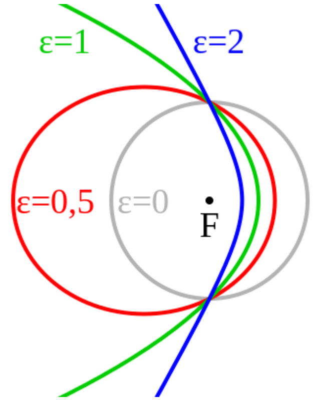
Credit:https://en.wikipedia.org/wiki/Two-body_problem

$$\ddot{\vec{x}}_1 = -Gm_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

$$\ddot{\vec{x}}_2 = -Gm_1 \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_1 - \vec{x}_2|^3}$$



$$r(\nu) = \frac{p}{1 + \varepsilon \cdot \cos\nu}$$



  - Jacobi coordinates

$$\vec{r} := \vec{x}_1 - \vec{x}_2$$

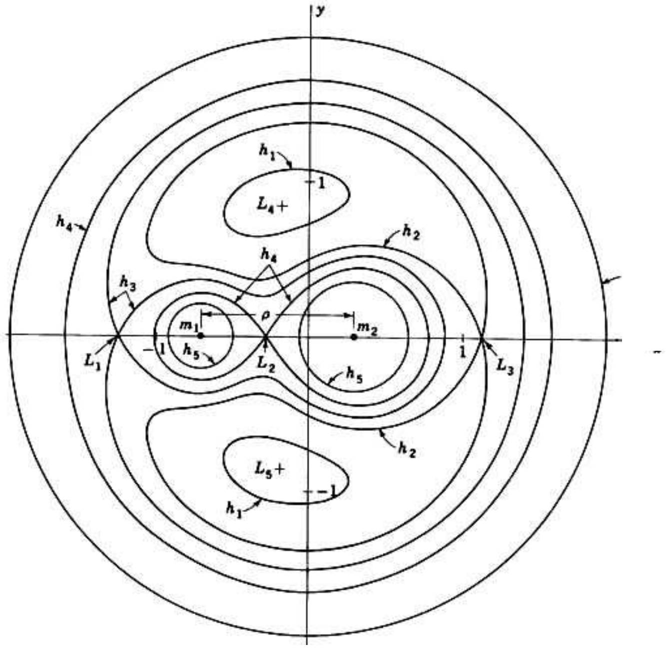
$$\vec{R} := (m_1\vec{x}_1 + m_2\vec{x}_2)/(m_1 + m_2)$$

$$\ddot{\vec{R}} = 0,$$

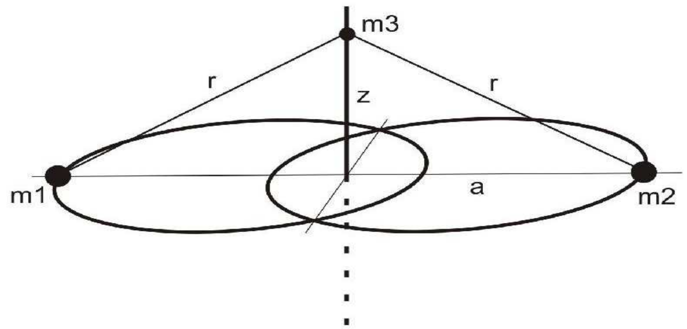
$$\ddot{\vec{r}} = -GM\frac{\vec{r}}{|\vec{r}|^3}$$

  - Integrals of motion: Energy: $E = \frac{\mu}{2}\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r}$, Angular momentum: $L = \mu r^2 \dot{\phi}$

Introduction



- 3-Body problem:
  - No general, analytic solution
  - However, special solutions do exist:
    - Euler solution (1767): collinear aligned masses (3 solution families)
    - Lagrange solution (1772): masses form equilateral angle (2 solution families)
    - => these solutions become Lagrange points in the Restricted 3-Body problem
    - Sundman solution (Karl Frithiof Sundman, 1909): (extremely slowly) convergent infinite power series, practically one needs $10^{8\,000\,000}$ (!!) terms (Beloriszky, 1930)
  - Restricted 3-Body problem ($m_P \sim 0$):
    - MacMillan / Sitnikov problem



- N-Body problem:
  - No general, analytical solution
  - Q. Wang (1991): Generalization of Sundman solution

Introduction – Chaos Theory

- Chaos theory

Credit: https://en.wikipedia.org/wiki/Chaos_theory

  - Studies behavior of dynamical systems
  - Small differences in initial conditions
    -> widely diverging outcomes
  - => long term prediction nearly impossible in general
  - Systems are deterministic -> ‘Deterministic Chaos’
  - Future fully determined by initial conditions -> no random elements
  - “When the present determines the future, but the approximate present does not approximately determine the future”
  - Applications: meteorology, sociology, physics, computer science, engineering, economics, biology, ecology, philosophy, …

Introduction – Chaos Theory

- Definition of Chaos:

  – Common usage (non-scientific): “chaos” means “a state of disorder”

  – definition in Poincaré sense: dynamical behavior is not quasi-periodic

  -> does not necessarily mean that system will disintegrate during any limited period of time (-> solar system)

  – *Stability* in Poisson sense: stability is related to the preservation of a certain neighborhood relative to the initial position of the trajectory

  -> in conservative systems, quasi periodic orbits remain always confined within certain limits, in this sense they are stable

Introduction – Chaos Theory

- Definition of Chaos:

  – no universally accepted mathematical definition of chaos exists

  – Mathematical definition by Robert L. Devaney (1989):

    – to classify a dynamical system, i.e. map $f: X \rightarrow X$, as chaotic, it must fulfil these properties:

      1. f must be sensitive to initial conditions
      2. f must have topological mixing
      3. the set of periodic orbits of f is dense in X

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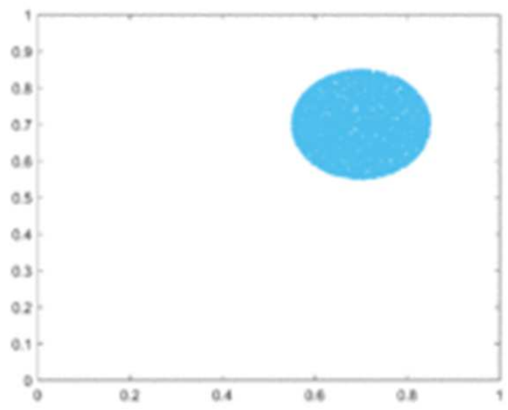


Six iterations of map $x_{k+1} = 4x_k (1- x_k)$

Chaos Theory - Systems

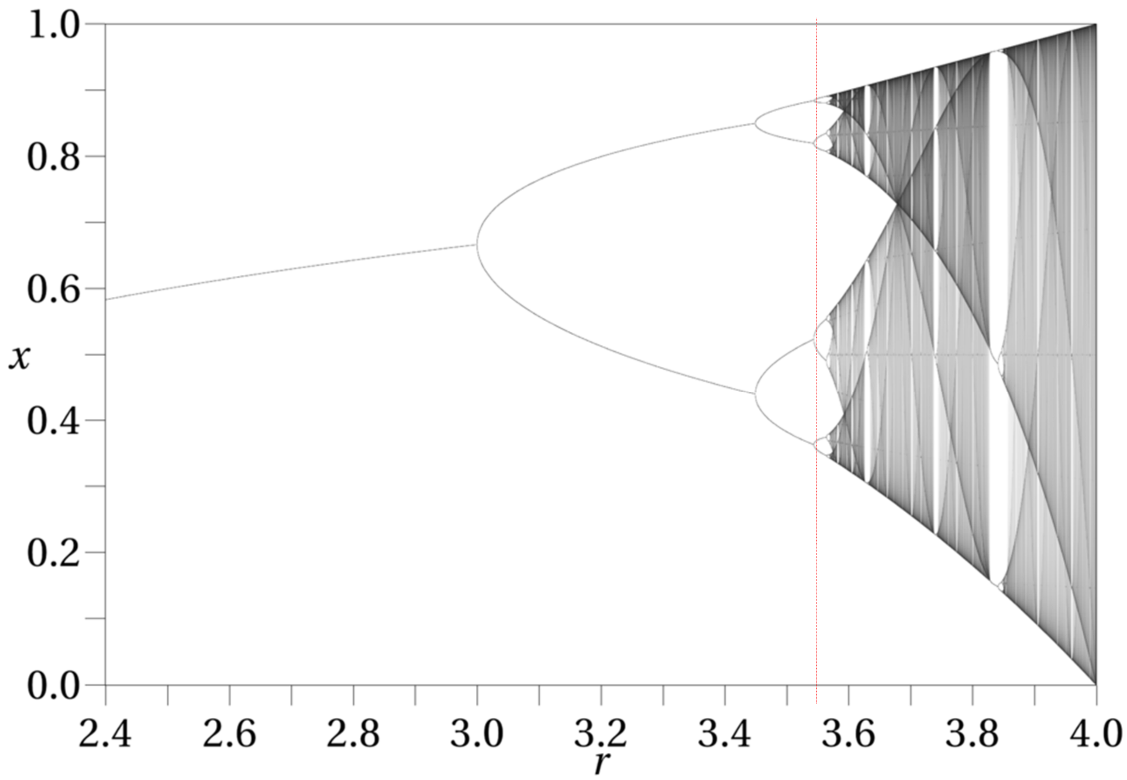
- Logistic Map

$$x_{n+1} = F(x) = rx_n(1 - x_n)$$

  - Fix point (order k):

$$x_0 = \underbrace{F(F(F(F(\ldots F(x_0)\ldots))))}_{\text{k-times}}$$

  - Map becomes chaotic for $r >= r_\infty = 3.5699\ldots$



Credit: https://en.wikipedia.org/wiki/Logistic_map

$r_\infty = 3.5699\ldots$

Chaos Theory - Systems

- Standard Map

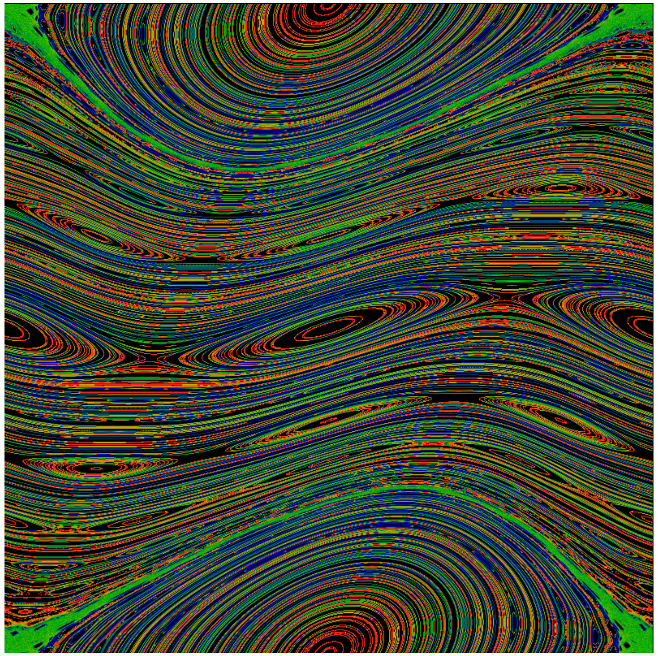
$$I_{n+1} = I_n + K \sin \theta_n$$
$$\theta_{n+1} = \theta_n + I_{n+1}$$

  – Surface of Section

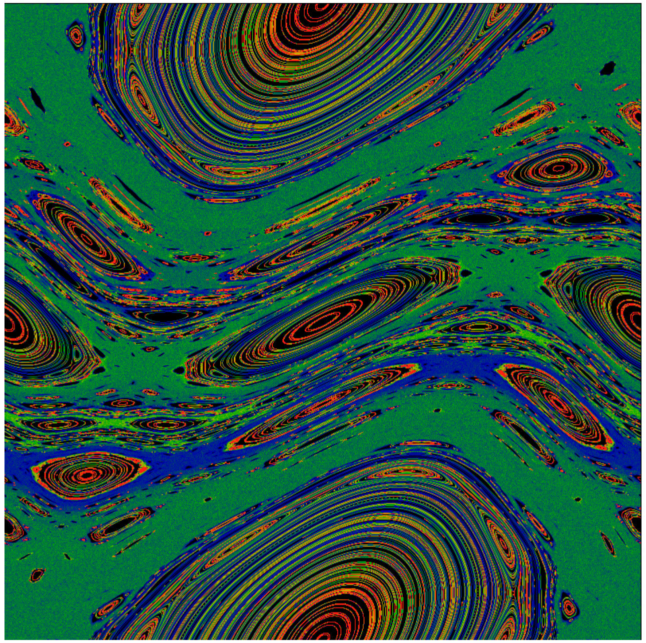
Chaos sets in at K ~ 0.9716535…

(Golden KAM-Torus)

Phase Space (θ, I) for K = 0.6



Phase Space (θ, I) for K = 0.971635



Credit: https://en.wikipedia.org/wiki/Standard_map

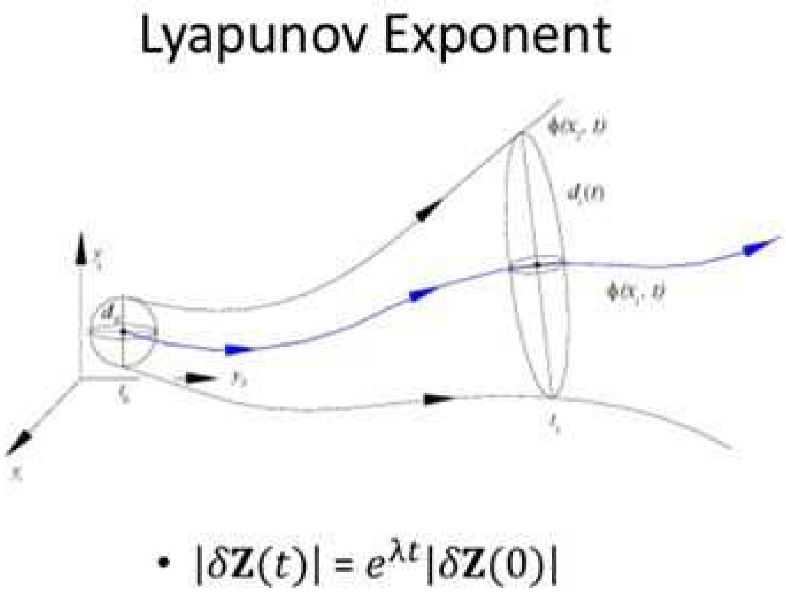
Chaos Theory – Lyapunov Exponent

- Lyapunov Characteristic Exponent (LCE)

  - is a quantity of a dynamical system that characterizes the rate of separation of infinitesimally close trajectories
  - Two trajectories in phase space with initial separation $\delta \mathbf{Z}_0$ diverge at a rate given by

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}_0|$$

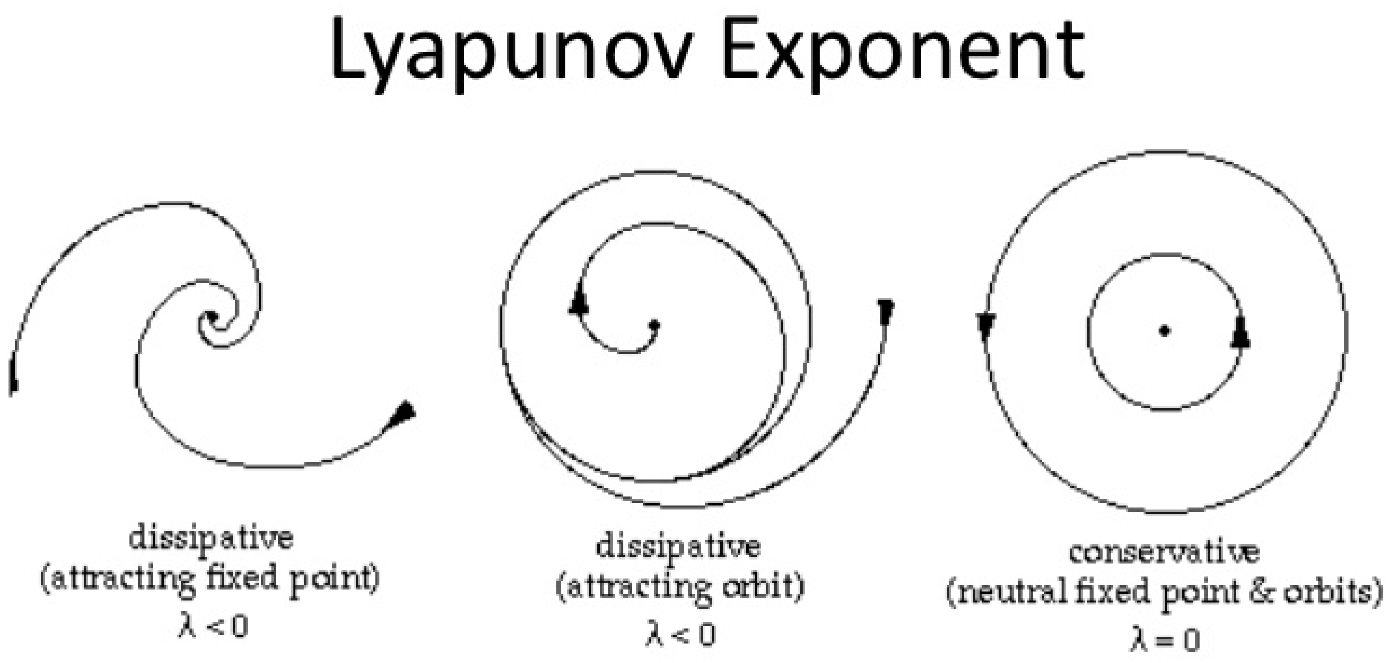
  where λ is the Lyapunov exponent



Credit: http://www.slideshare.net/dvidby0/lyapunov-exponent-of-time-series-data

  - generally, rate of separation is different for different orientations of initial separation vector
  - there is a spectrum of Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_N$, equal in number to dim of phase space -> largest λ_k … Maximal Lyapunov exponent (MLE)
  - Positive MLE is an indication that the system is chaotic

Chaos Theory – Lyapunov Exponent



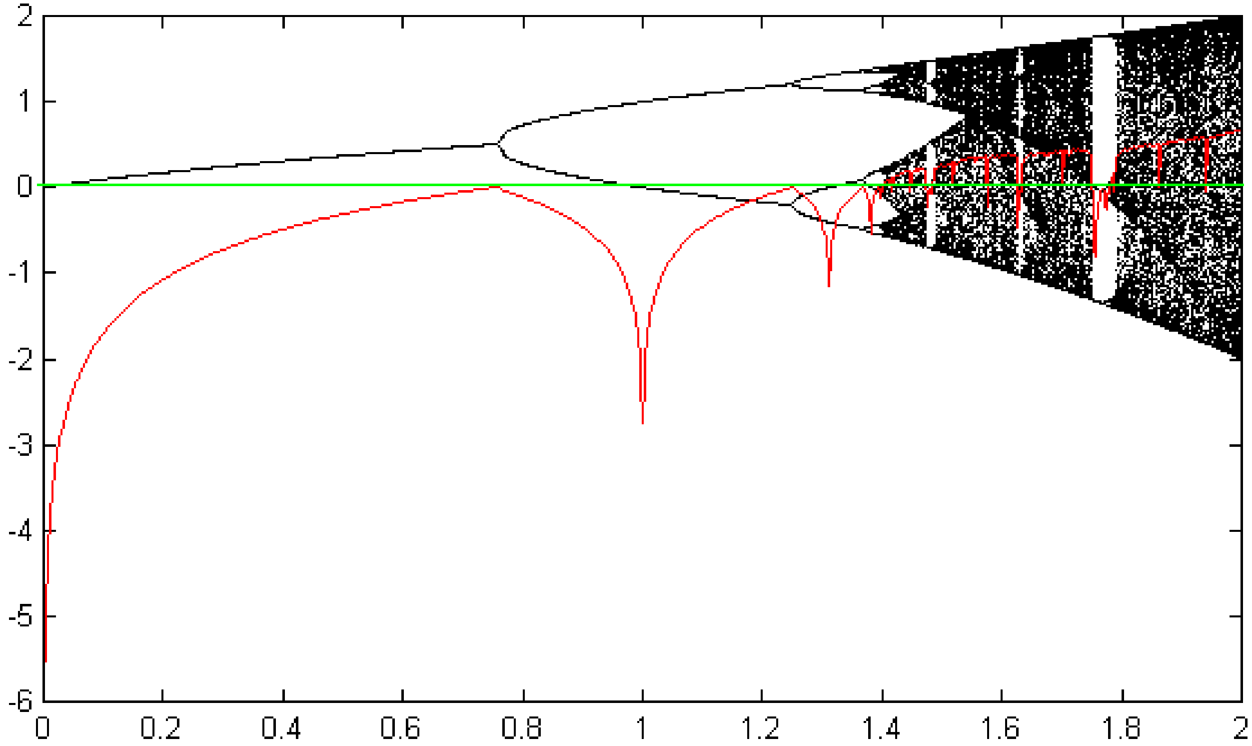
Credit: http://www.slideshare.net/dvidby0/lyapunov-exponent-of-time-series-data

- Maximal Lyapunov Exponent

$$\lambda = \lim_{t \to \infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

Chaos Theory – Lyapunov Exponent

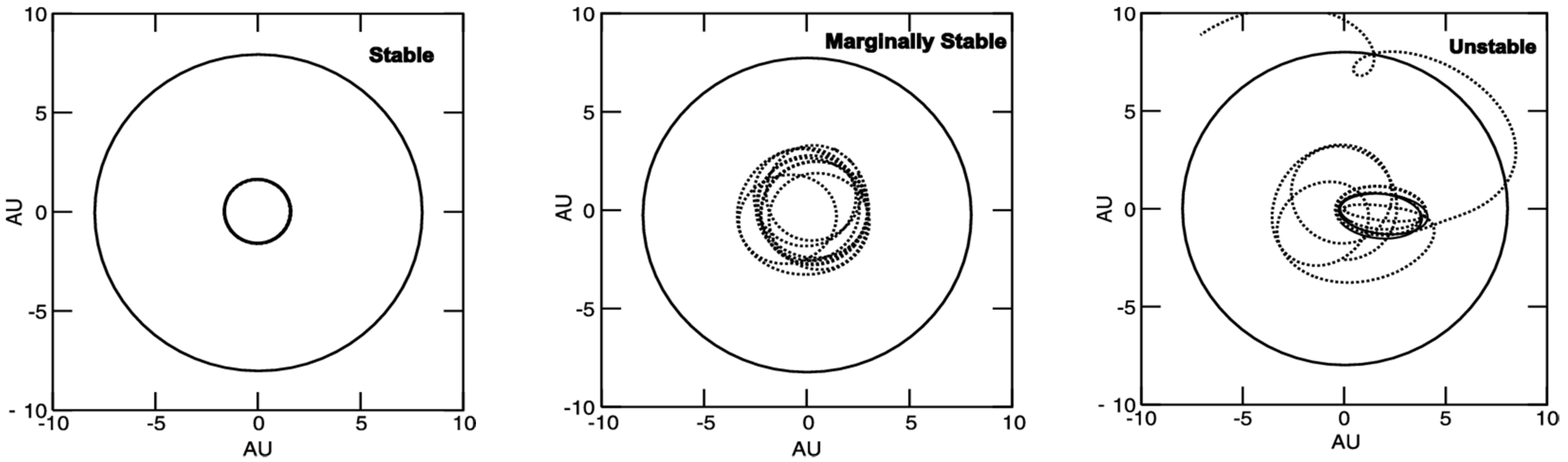
- Lyapunov Exponent for Logistic Map



Credit: http://math.arizona.edu/~ura-reports/001/huang.pojen/2000_Report.html

Stability of Planet orbit

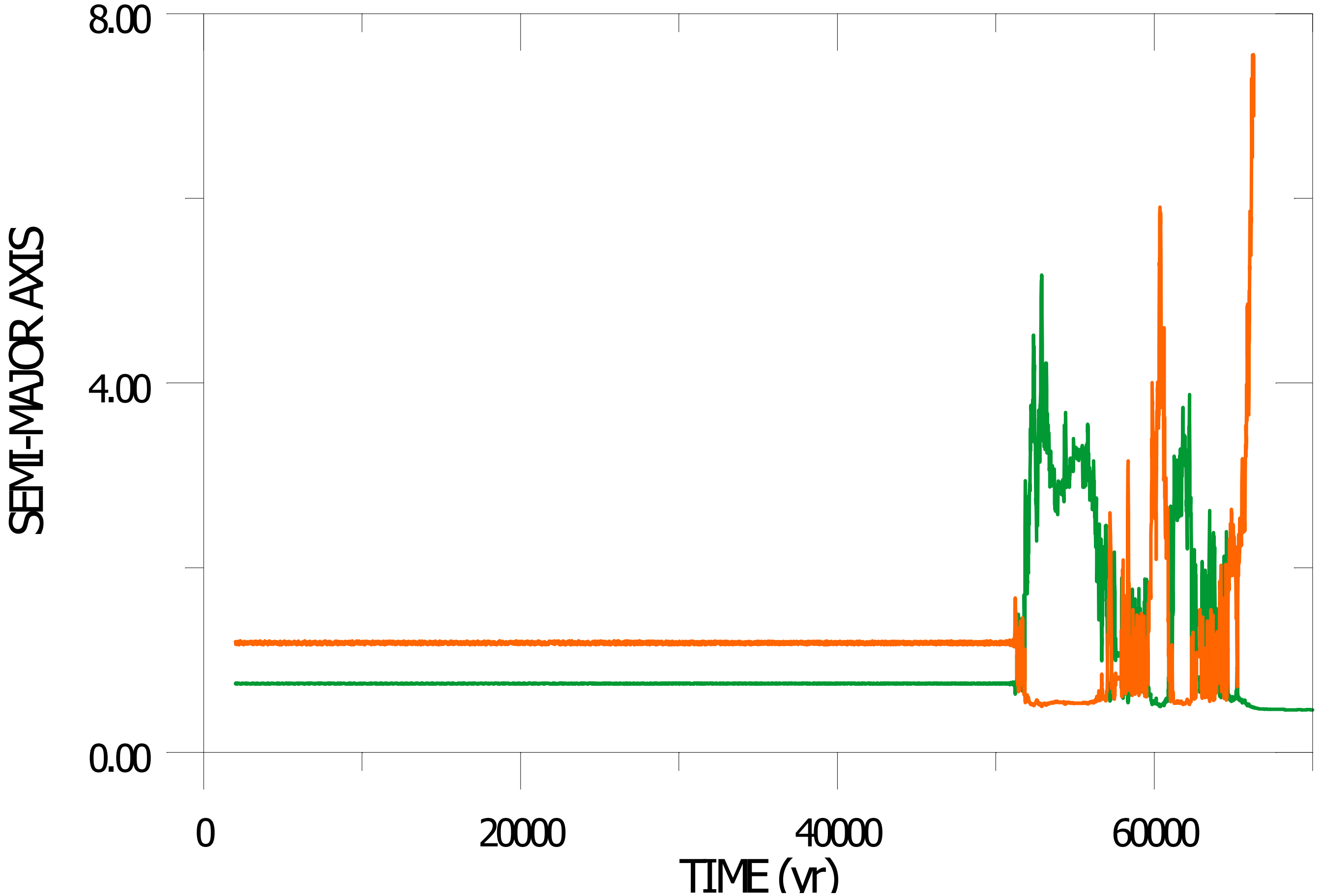
- Stability of planetary orbits in binary systems (Z. E. Musielak, et al., 2005)



Credit: http://www.aanda.org/articles/aa/full/2005/16/aa0238-04/aa0238-04.right.html

Stability of Planet orbit

- Evolution of semimajor axes of HD 82943c (S. Ferraz-Mello, 2004)



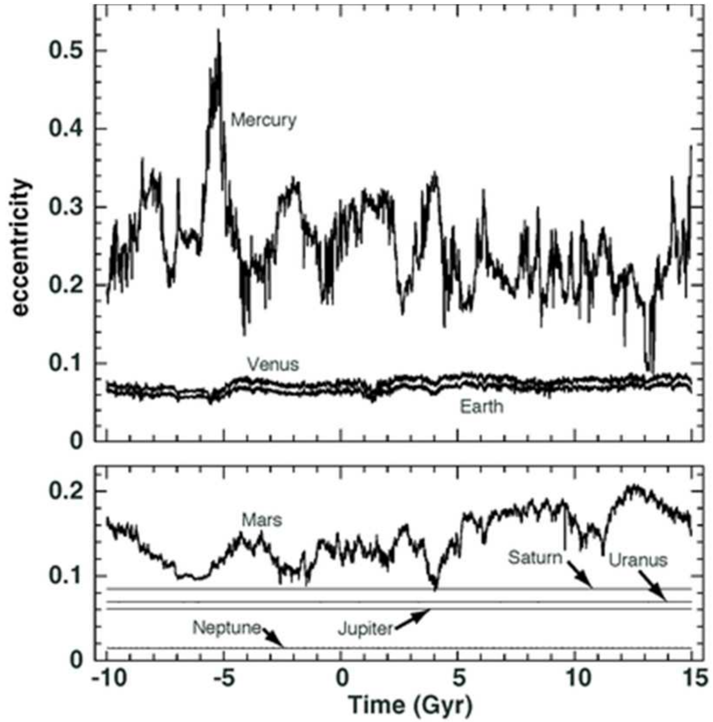
Stability of Solar system

- History (http://www.scholarpedia.org/article/Stability_of_the_solar_system)
  - Hipparchus, Ptolemy: epicycles
  - Copernicus, Kepler (laws: 1609-1618)
  - Newton’s gravitation law 1687
  - Laplace-Lagrange
    - Correctly formulated equations of motion

      -> Perturbation theory
  - Hamilton, Jacobi, Poincaré: 1892-1899 not possible to integrate equations of motion of 3-body problem
  - Kolmogorov, Arnold, Moser (KAM theorem), 1950-60: if masses, eccentricities, inclinations of planets are small enough -> many initial conditions lead to quasiperiodic trajectories, actual masses of planets are much too large to apply directly to solar system (Michel Hénon computed that masses needs to be smaller than 10^{-320})

Stability of Solar system

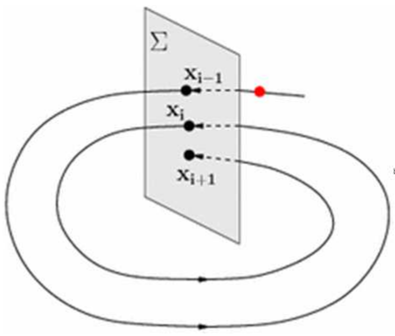
- Chaos in the Solar system:
  - Integration over 200 Mio years (Laskar, 1989) showed that solar system is in principle chaotic, with Lyapunov time of about 50 Mio years
  - Marginal stability:
    - Solar system is full
    - Resonances important (MMR, Secular)
    - 3-5 billion years to allow collision
    - Solar system is in principal unstable
    - But catastrophe time-scale is 5 billion year

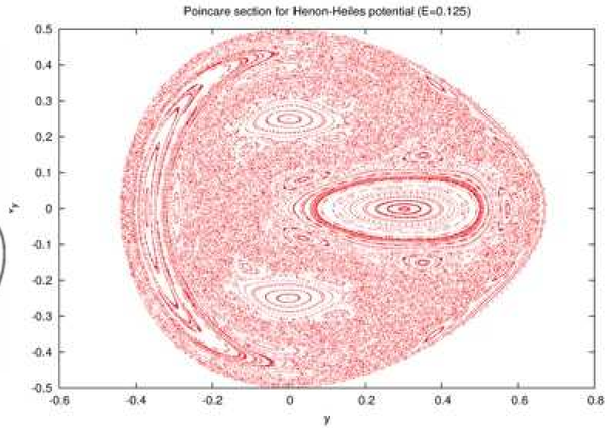


Chaos Indicators - Overview

- Chaos Indicators are techniques to detect chaos (not to proof chaos)
- Indicators versus Order Parameter (e.g. magnetization M in ferromagnetism)

- "Slow" techniques:
  - Poincaré Surfaces of Section
    - Poincaré, Birkhoff, Hénon & Heiles
    - 2 degrees of freedom
    - costly numerical integrations
    - in some cases it is impossible to obtain a transverse section for the whole flow





  - Maximum Lyapunov Exponent (MLE)
    - long time integration

$$\lambda = \lim_{t \to \infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

Chaos Indicators – Fast Techniques

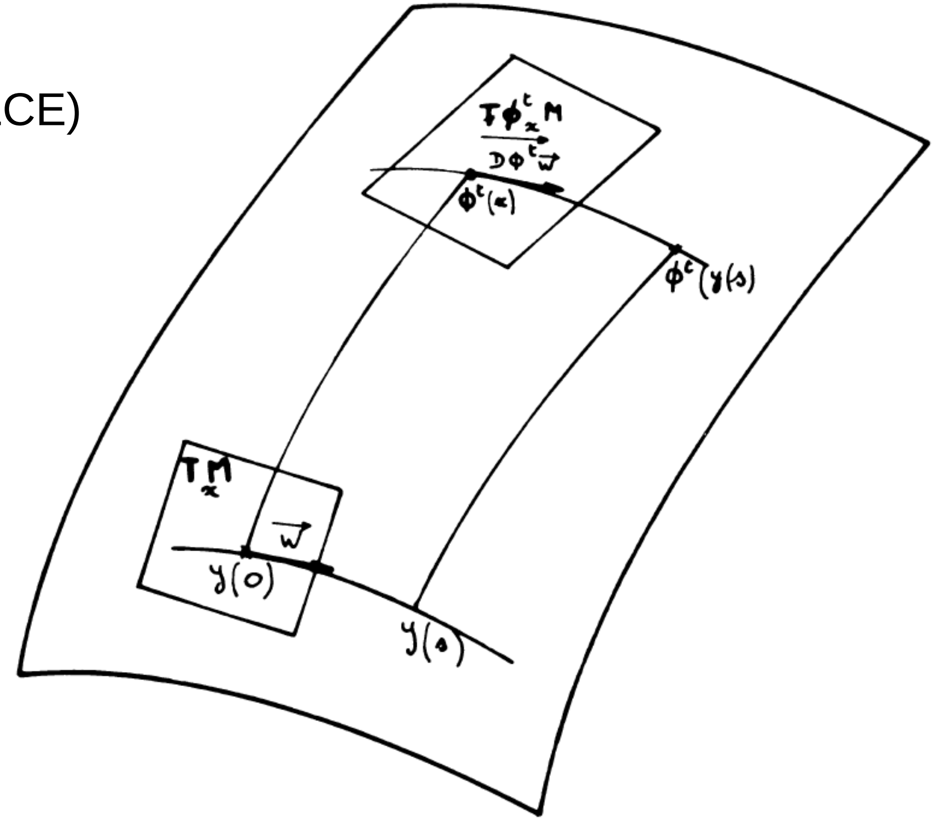
- FLI: "Fast Lyapunov Indicator" (C. Froeschlé, 1984; C. Froeschlé, E. Lega, R. Gonczi, M, Guzzo, 1997-2015)
- OFLI / OFLI2: "Orthogonal Fast Lyapunov Indicator" (M. Fouchard, C. Froeschlé, E.Lega, 2002; R. Barrio, P.M. Cincotta)
- MEGNO: "Mean Exponential Growth of Nearby Orbits" (P.M. Cincotta, C. Simó, 2000; C.M. Giordano, N. Maffione)
- SALI / GALI: "Smaller/Generalized Alignment Indices" (C.H. Skokos, 2001; T. Manos)
- RLI: "Relative Lyapunov Indicators" (Z. Sándor, *et al.* 2004, N. Maffione)

Fast Lyapunov Indicator (FLI)

- FLI introduced by C. Froeschlé, *et al*. (1997)
- Based on Lyapunov Characteristic Exponent (LCE)
  - Exponential-like divergence of originally nearby trajectories
- Easy (to implement) and sensitive tool for detection of chaos (“Arnold web”)

$$w = \frac{\partial y}{\partial s}\Big|_{s=0} \qquad T_x M$$

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{||D\,\phi_x^t\, w||}{||\,w\,||} = \chi(x,w)$$

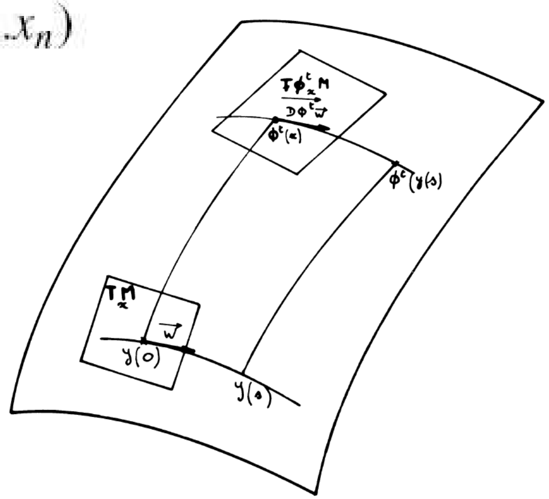


Credit: C. Froeschlé, 1984

Fast Lyapunov Indicator (FLI)

- Definition of Fast Lyapunov Indicator

Credit: C. Froeschlé, 1984



  - Set of differential equations: $\frac{dx}{dt} = F(x) \;,\; x = (x_1, x_2, ...x_n)$

  - Equations of motion:

$$\frac{dx}{dt} = F(x)$$
$$\frac{dv}{dt} = \frac{\partial F}{\partial x} v.$$

  - Evolution of tangent vector v:

$$x(t+1) = \psi(x(t))$$
$$v(t+1) = \frac{\partial \psi}{\partial x}(x(t))v(t).$$

=> Fast Lyapunov indicator: $\mathrm{FLI}_t(x(0), v(0)) = \log \frac{\|v(t)\|}{\|v(0)\|}$

improved version (reduce fluctuations): $\mathrm{FLI}(x(0), v(0), t) = \sup_{0 \le k \le t} \log ||v(k)||$

Fast Lyapunov Indicator (FLI)

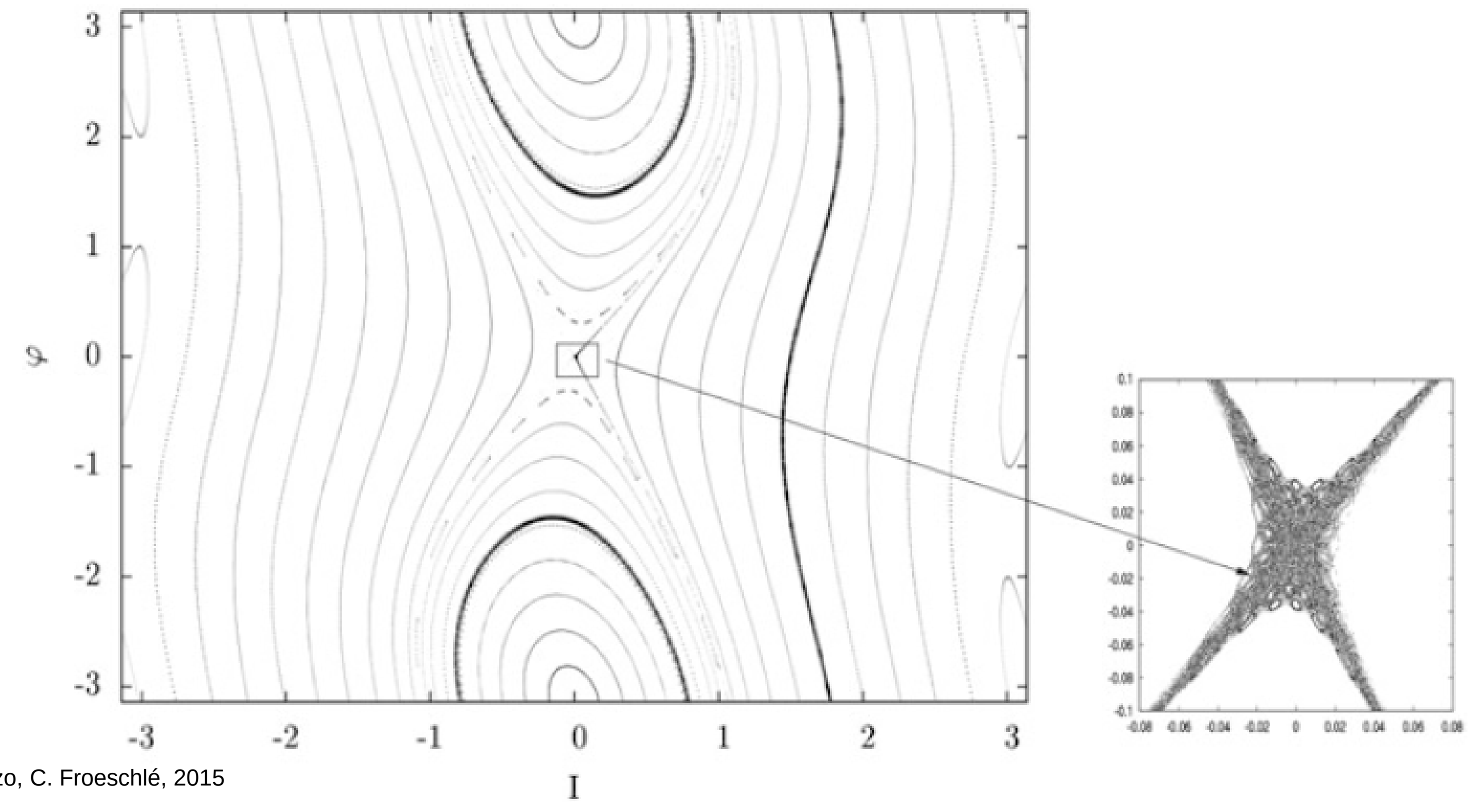
- Properties of FLI
  - Quantity $\mathrm{FLI}_t(x(0), v(0))/t$ tends to the largest Lyapunov exponent (of spectrum of LCEs) as t goes to infinity
  - If differential equations are Hamiltonian and if motion is regular -> largest Lyapunov exponent is zero, otherwise it is positive. This property is largely used to discriminate between chaotic and ordered motions
  - However, among regular motions the ordinary Lyapunov exponent does not distinguish between circulation and libration orbits.
  - In contrast, the FLI distinguish between them
  - How to choose v(0) for practical implementation? Special choices of v(0) have to be avoided
  - => compute average (or alternatively the maximum) of the FLIs obtained for an orthonormal basis of tangent vectors

Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map $(I(t+1), \varphi(t+1)) = \psi(I(t), \varphi(t))$ $(I, \varphi) \in \mathbb{R} \times \mathbb{S}^1$

$$\psi(I, \varphi) = (I + \epsilon \sin(\varphi + I), \varphi + I)$$

- 3 orbits
  - Libration orbit
  - Circulation orbit
  - Chaotic orbit



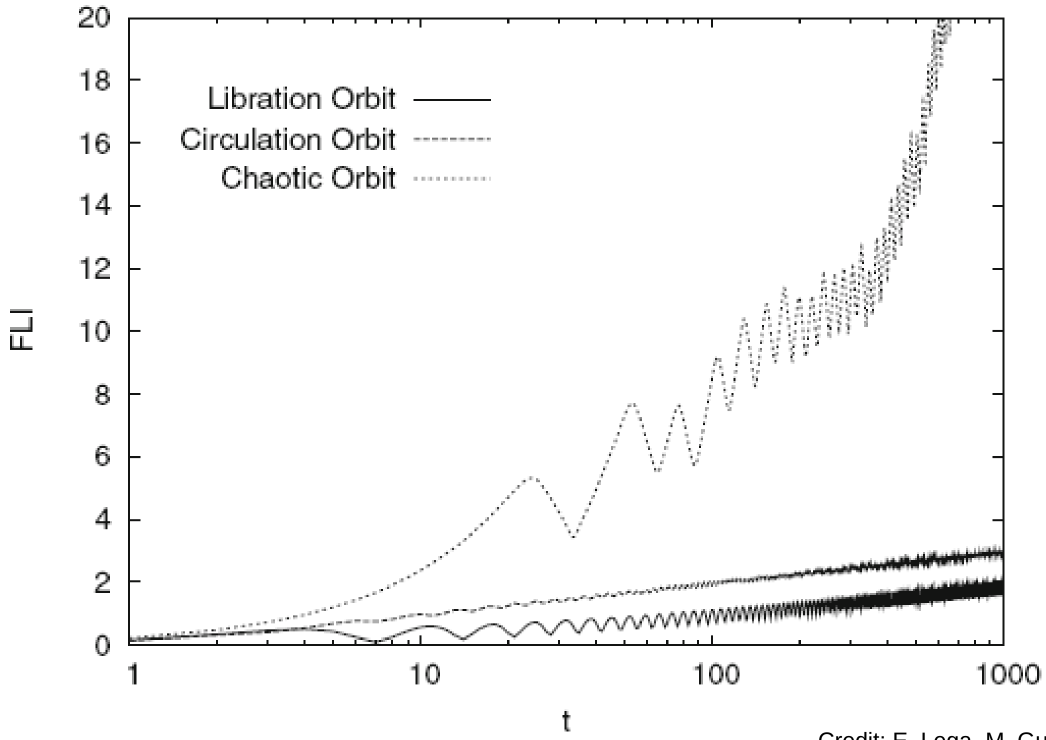
Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

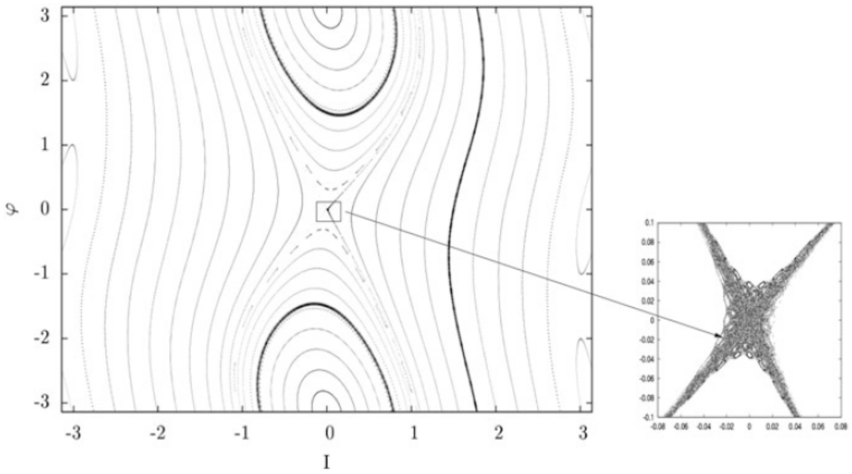
Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

$$\mathrm{FLI}_t(x(0), v(0)) = \log \frac{\|v(t)\|}{\|v(0)\|}$$

Time evolution of FLI





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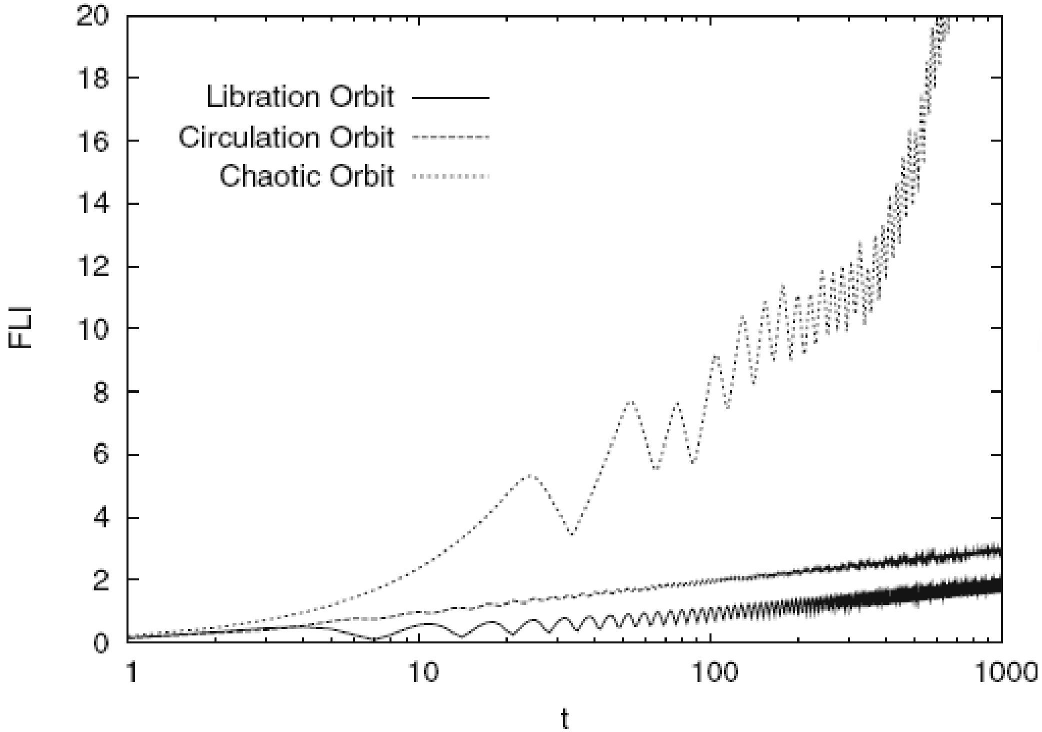
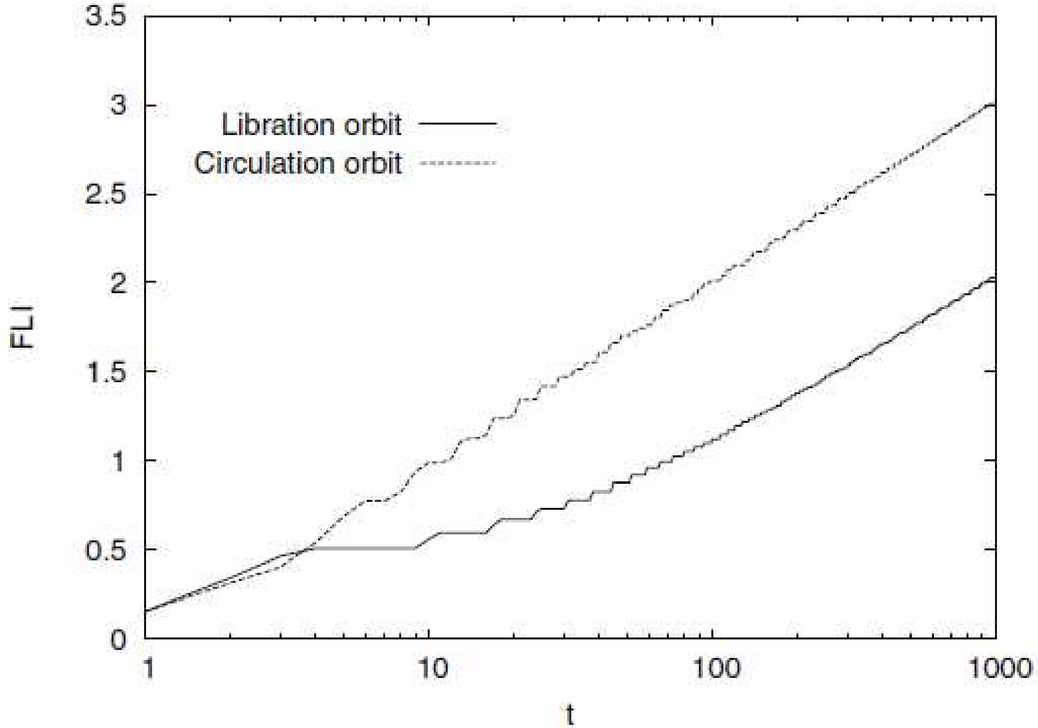


$$\mathrm{FLI}(x(0), v(0), t) = \sup_{0 \le k \le t} \log ||v(k)||$$



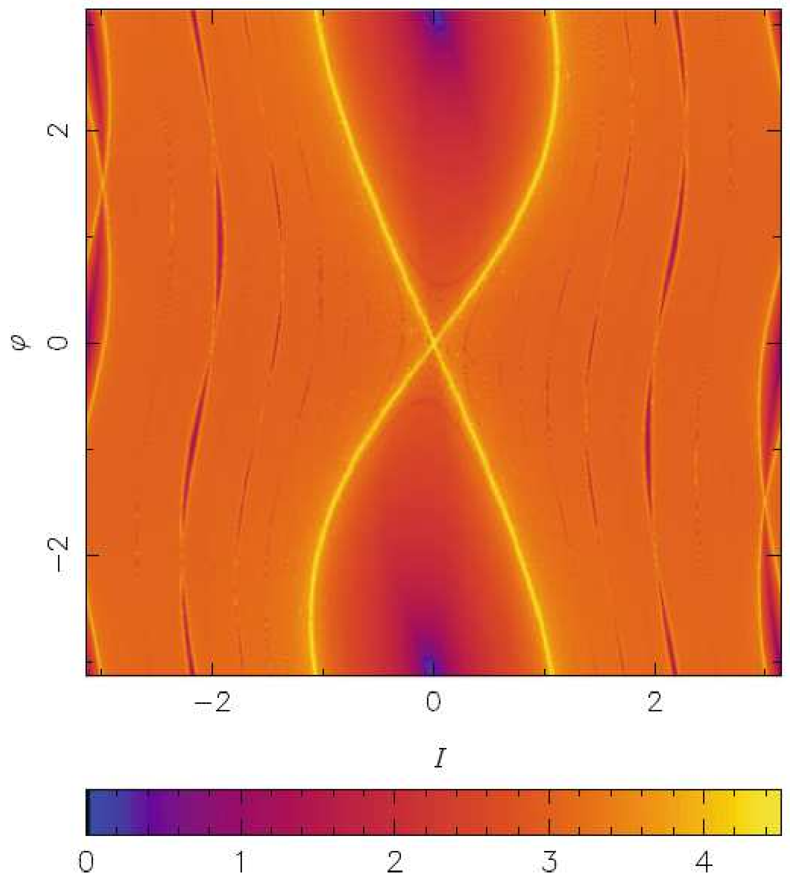
Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

$$\mathrm{FLI}(x(0), v(0), t) = \sup_{0 \le k \le t} \log ||v(k)||$$

  - FLI for t = 1000
  - Grid of 900 x 900 initial conditions
  - 2 orthogonal initial vectors
    - v(0) = (1,0), w(0) = (0,1)
    - Largest FLI is plotted



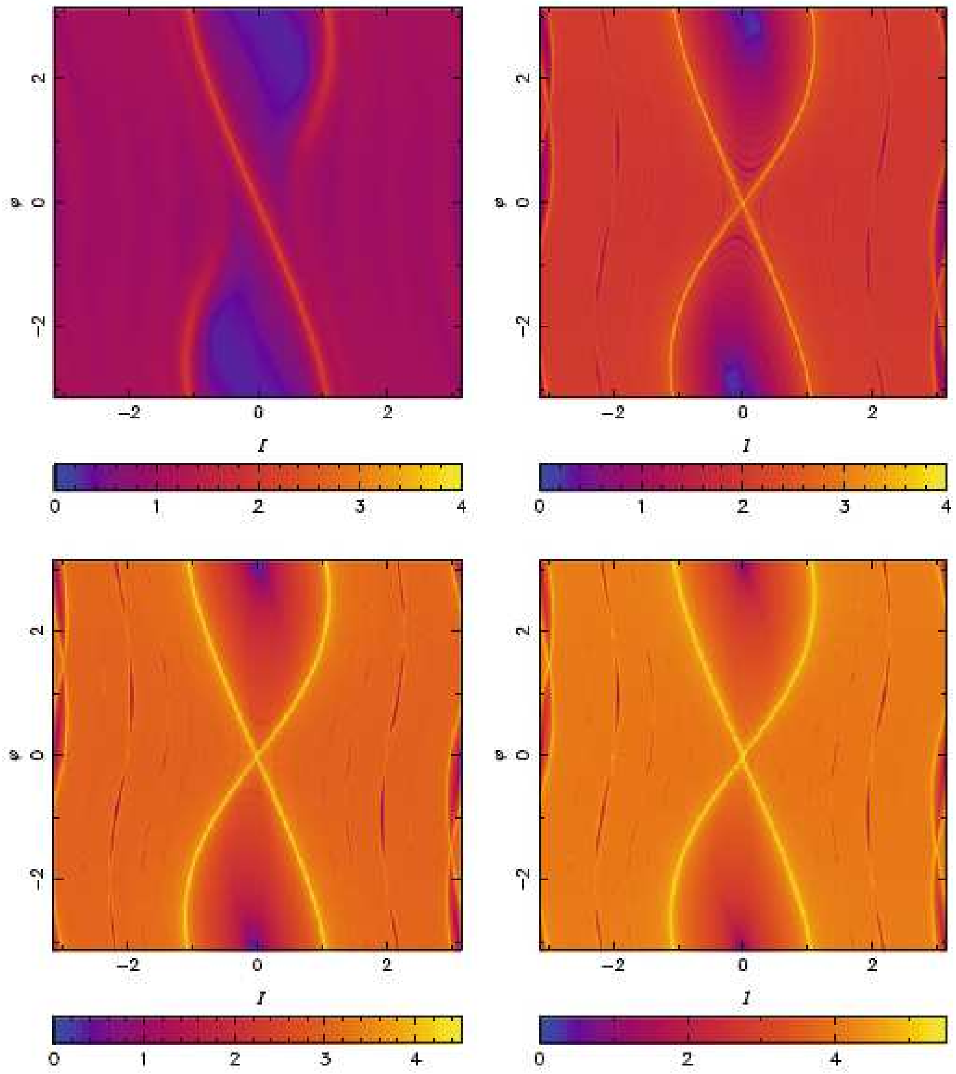
Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

Fast Lyapunov Indicator (FLI)

- FLI applied on Standard Map

- Choice of integration time:
  - t = 10 (top left), t = 100 (top right)
  - t = 1000 (bottom left), t = 10 000 (bottom right)

- t = 1000 seems to be appropriate in that example

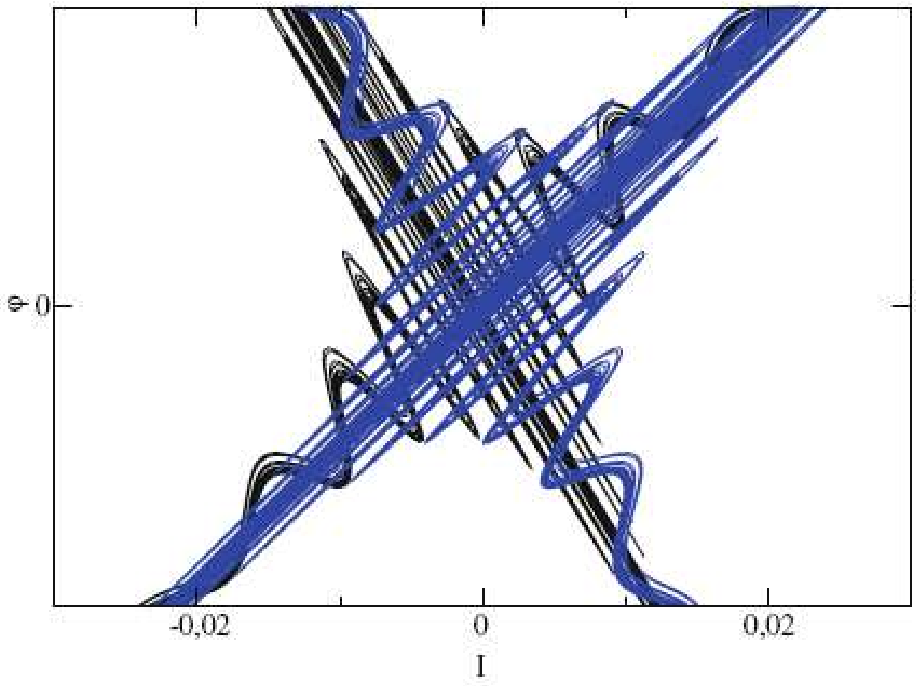


Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

Fast Lyapunov Indicator (FLI)

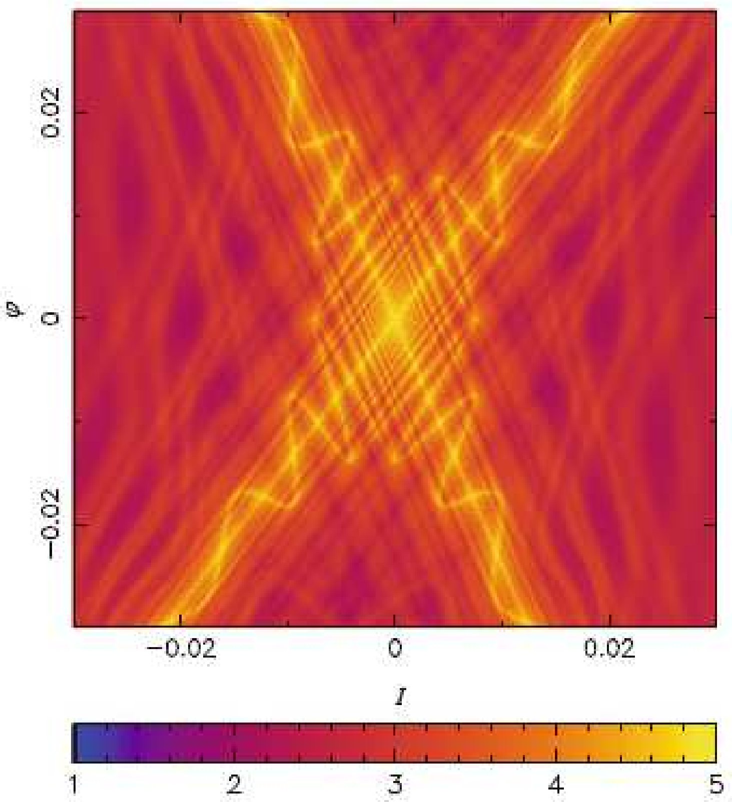
- FLI applied on Standard Map

Detection of regular/chaotic regions via method of set propagation -> rather costly



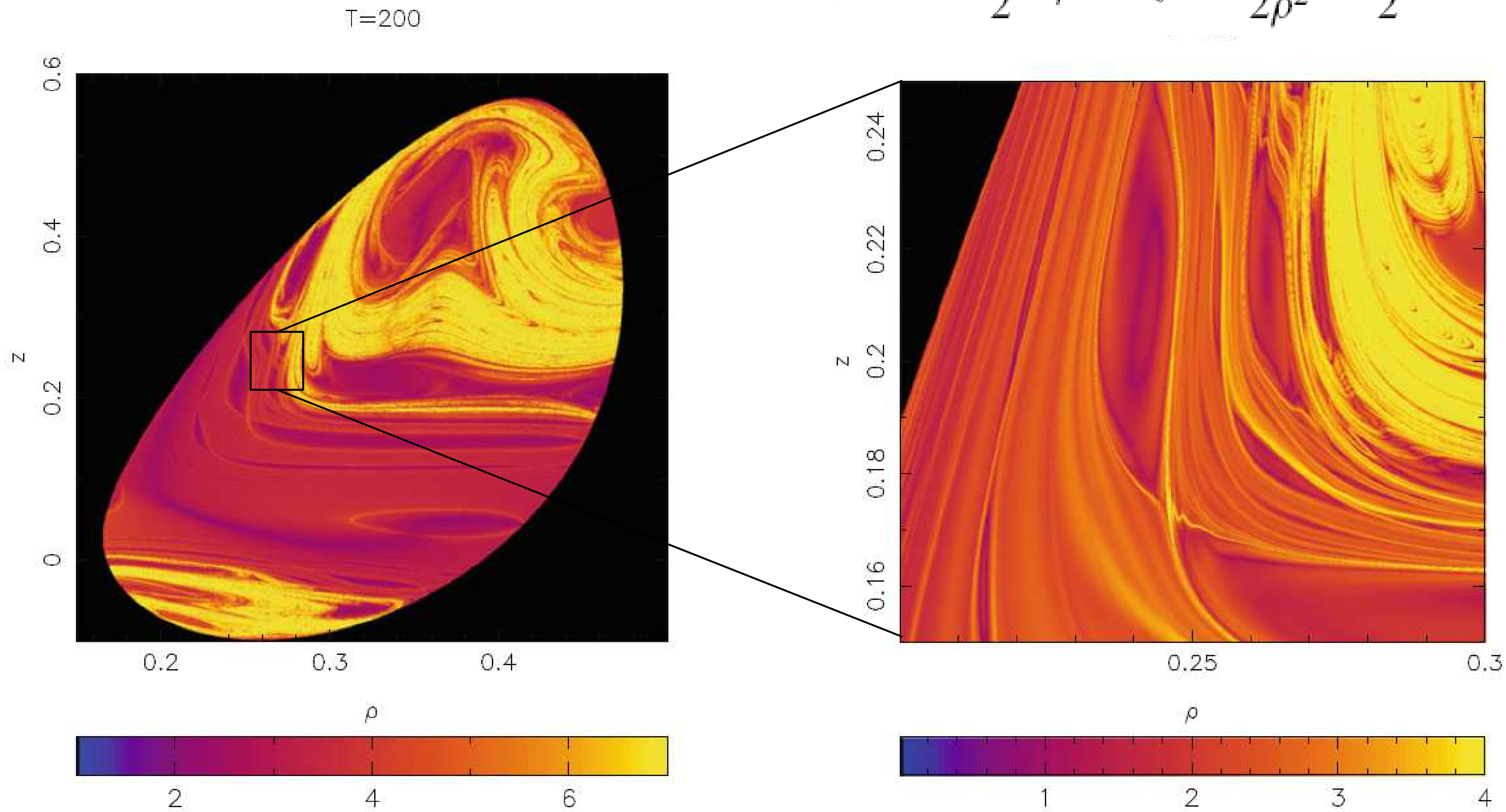
Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

FLI with t = 50 -> very efficient



Fast Lyapunov Indicator (FLI)

- FLI applied to Continuous System: $$H(\rho, z, p_\rho, p_z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{h_z^2}{2\rho^2} + \frac{1}{2}\log(\rho^2 + z^2) - z$$



Credit: E. Lega, M. Guzzo, C. Froeschlé, 2015

Orthogonal Fast Lyapunov Indicator (OFLI)

- OFLI introduced by M. Fouchard, C. Froeschlé, E. Lega, 2002

- in case of OFLI one takes component orthogonal to flow

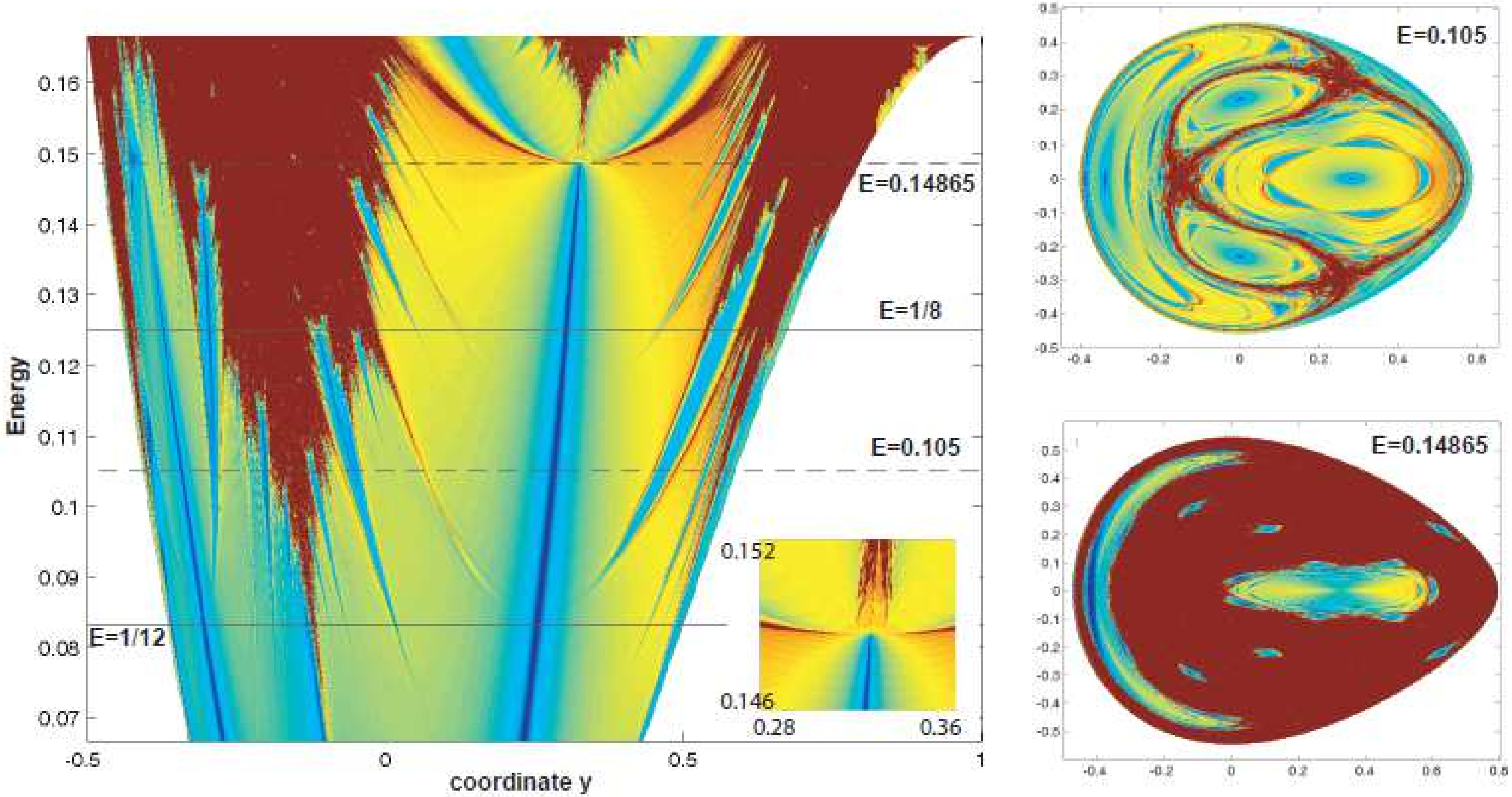
$$\mathrm{FLI}(\mathbf{y}(0), \delta\mathbf{y}(0),\ t_f) = \sup_{0<t<t_f} \log \|\delta\mathbf{y}(t)\|$$

$$\mathrm{OFLI}(\mathbf{y}(0), \delta\mathbf{y}(0),\ t_f) = \sup_{0<t<t_f} \log \|\delta\mathbf{y}^{\perp}(t)\|$$

- with OFLI one can distinguish between periodicity among the regular component
- OFLI tends to a constant value for a periodic orbit
- for quasiperiodic and chaotic motion same behavior as FLI

FLI Example – Hénon-Heiles system

$$\mathcal{H}(x, y, X, Y) = \frac{1}{2}(X^2 + Y^2 + x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$



MEGNO

- MEGNO ("Mean Exponential Growth of Nearby Orbits") (P.M. Cincotta, C. Simó, 2000)

  - Suitable fast indicator to separate regular from chaotic motion
  - Provides relevant information of global dynamics and the fine structure of phase space
  - Yields good estimate of the LCN with a comparatively small computational effort
  - Provides clear picture of resonance structures, location of stable and unstable periodic orbits, as well as measure of rate of divergence of unstable orbits
  - Feasible to investigate nature of orbits that have small, positive Lyapunov number
  - Converges to null value of LCN faster than classical algorithm to compute the LCN

MEGNO

- Definition of MEGNO Indicator

  – N-dim Hamiltonian $H(\mathbf{p},\mathbf{q})$ with $\mathbf{p},\ \mathbf{q} \in \mathbb{R}^N$

  $\mathbf{x} = (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2N},\ \mathbf{v} = (-\partial H/\partial \mathbf{q},\ \partial H/\partial \mathbf{p}) \in \mathbb{R}^{2N}$

  – Equations of motion: $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$

  – $\gamma(\mathbf{x_0};t)$: arc of an orbit of the flow over compact energy surface

  $M_h \subset \mathbb{R}^{2N},\ M_h = \{\mathbf{x} : H(\mathbf{p},\mathbf{q}) = h\} \qquad h = constant$

  $\gamma(\mathbf{x_0};t) = \{\mathbf{x}(t';\mathbf{x_0}) : \mathbf{x_0} \in M_h,\ 0 \le t' < t\}$

  – Largest Lyapunov Characteristic Number (LCN)

$$\sigma(\gamma) = \lim_{t\to\infty} \sigma_1(\gamma(\mathbf{x_0};t)), \quad \sigma_1(\gamma(\mathbf{x_0};t)) = \frac{1}{t}\ln\left[\|\vec{\delta}\gamma(\mathbf{x_0};t)\|\right]$$

$$\sigma(\gamma) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\dot{\delta}\gamma(\mathbf{x_0};t')}{\delta\gamma(\mathbf{x_0};t')}\mathrm{d}t' = \overline{\left(\dot{\delta}/\delta\right)}$$

MEGNO

- Definition of MEGNO Indicator

$$\sigma(\gamma) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\dot{\delta\gamma}(\mathbf{x_0}; t')}{\delta\gamma(\mathbf{x_0}; t')} \mathrm{d}t' = \overline{\left(\dot{\delta}/\delta\right)}$$

$$\delta \equiv \|\vec{\delta}\|, \; \dot{\delta} \equiv \mathrm{d}\delta/\mathrm{d}t = \dot{\vec{\delta}} \cdot \vec{\delta}/\|\vec{\delta}\|$$

  - tangent vector $\vec{\delta}$ satisfies variational equation $\dot{\vec{\delta}} = \Lambda(\gamma(\mathbf{x_0}; t)) \cdot \vec{\delta},$ where Λ is the Jacobian matrix
  - Introduce MEGNO $Y(\gamma(\mathbf{x_0}; t))$
  - $Y(\gamma(\mathbf{x_0}; t)) = \frac{2}{t} \int_0^t \frac{\dot{\delta\gamma}(\mathbf{x_0}; t')}{\delta\gamma(\mathbf{x_0}; t')} t' \mathrm{d}t'$, average of $\ln\left[\delta\gamma(\mathbf{x_0}; t)/\delta\gamma(\mathbf{x_0}; t_0)\right] = \lambda t$

MEGNO

- Properties of MEGNO

  – for quasi-periodic orbits:

$$Y\left(\gamma_q(\mathbf{x_0};t)\right) \approx 2 - \frac{\ln(1+\lambda_q\,t)^2}{\lambda_q\,t} + O\left(\gamma_q(\mathbf{x_0};t)\right)$$

introducing time average

$$\overline{Y}(\gamma_q(\mathbf{x_0};t)) \equiv \frac{1}{t}\int_0^t Y(\gamma_q(\mathbf{x_0};t'))\mathrm{d}t'$$

gives

$$\overline{Y}\left(\gamma_q\right) \equiv \lim_{t\to\infty} \overline{Y}\left(\gamma_q(\mathbf{x_0};t)\right) = 2$$

MEGNO

- Properties of MEGNO

  – Asymptotic behavior:

$$\overline{Y}[\gamma(\mathbf{x_0};t)] \approx a_\gamma t + b_\gamma$$

  – Irregular, chaotic motion: $a_\gamma = \chi_\gamma/2 \text{ and } b_\gamma \approx 0$

  – Quasiperiodic motion: $a_\gamma = 0 \quad \text{and } b_\gamma \approx 2$

  – Stable periodic orbits (resonant elliptic tori): $b_\gamma \lesssim 2$

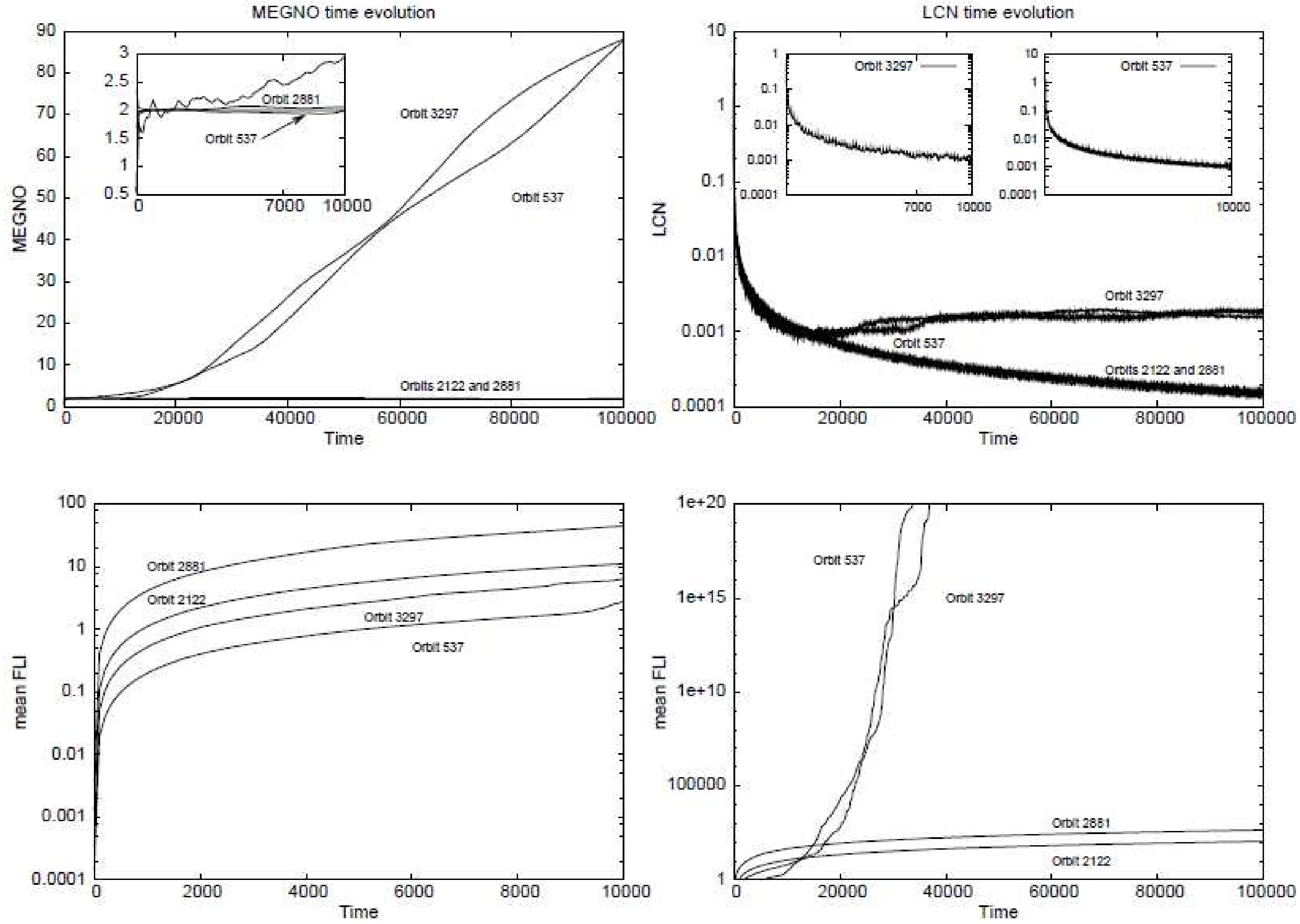
  – Unstable periodic orbits (hyperbolic tori): $b_\gamma \gtrsim 2$

MEGNO - Example

$$V(x,y,z) = -f_0(x,y,z) - f_x(x,y,z)\cdot(x^2-y^2) - f_z(x,y,z)\cdot(z^2-y^2)$$

$$f_n(x,y,z) = \frac{\alpha_n}{\left[p_n^{a_n} + \delta_n^{a_n}\right]^{\frac{ac_n}{a_n}}}$$

$$p_n^2 = x^2 + y^2 + z^2 + \epsilon^2$$



Credit: N. Maffione, et al., 2011

SALI / GALI

- Smaller/Generalized Alignment Index (SALI / GALI)
- introduced by H. Skokos (2001)

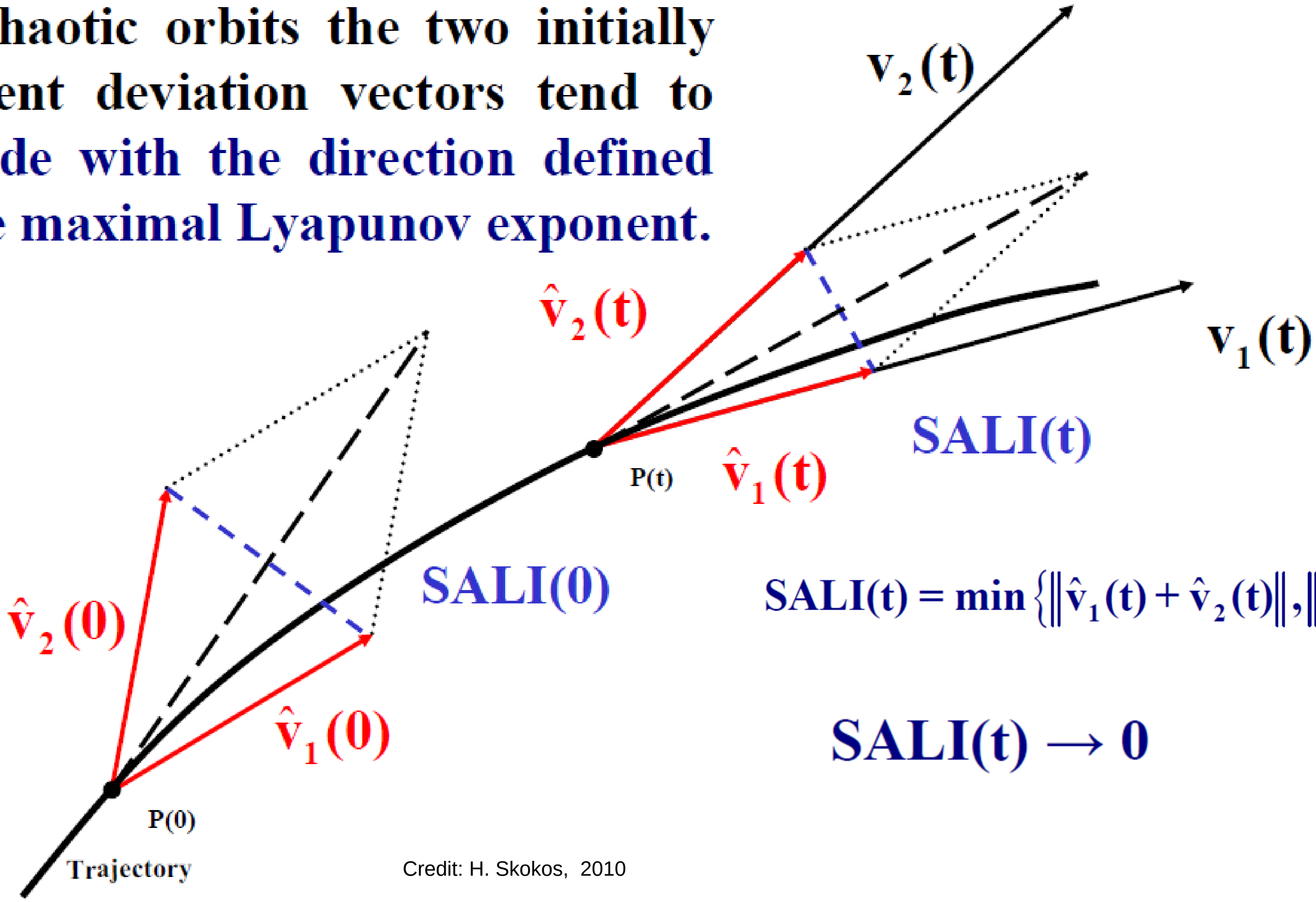
- Definition SALI
  - Orbit in n-dim space with initial condition: $P(0)=(x_1(0), x_2(0),\dots,x_n(0))$
  - Deviation vector: $v(0)=(dx_1(0), dx_2(0),\dots, dx_n(0))$

  - Evolution in time of two different deviation vectors $(v_1(0), v_2(0))$
  - Define SALI as:

$$\mathrm{SALI}(t) = \min\left\{\left\|\hat{v}_1(t) + \hat{v}_2(t)\right\|, \left\|\hat{v}_1(t) - \hat{v}_2(t)\right\|\right\} \qquad \hat{v}_1(t) = \frac{v_1(t)}{\left\|v_1(t)\right\|}$$

SALI / GALI

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent.



$$\mathrm{SALI}(t) = \min\left\{\left\|\hat{v}_1(t) + \hat{v}_2(t)\right\|, \left\|\hat{v}_1(t) - \hat{v}_2(t)\right\|\right\}$$

$$\mathrm{SALI}(t) \rightarrow 0$$

Credit: H. Skokos, 2010

SALI / GALI

- Properties of SALI
  - Behavior for chaotic orbits:

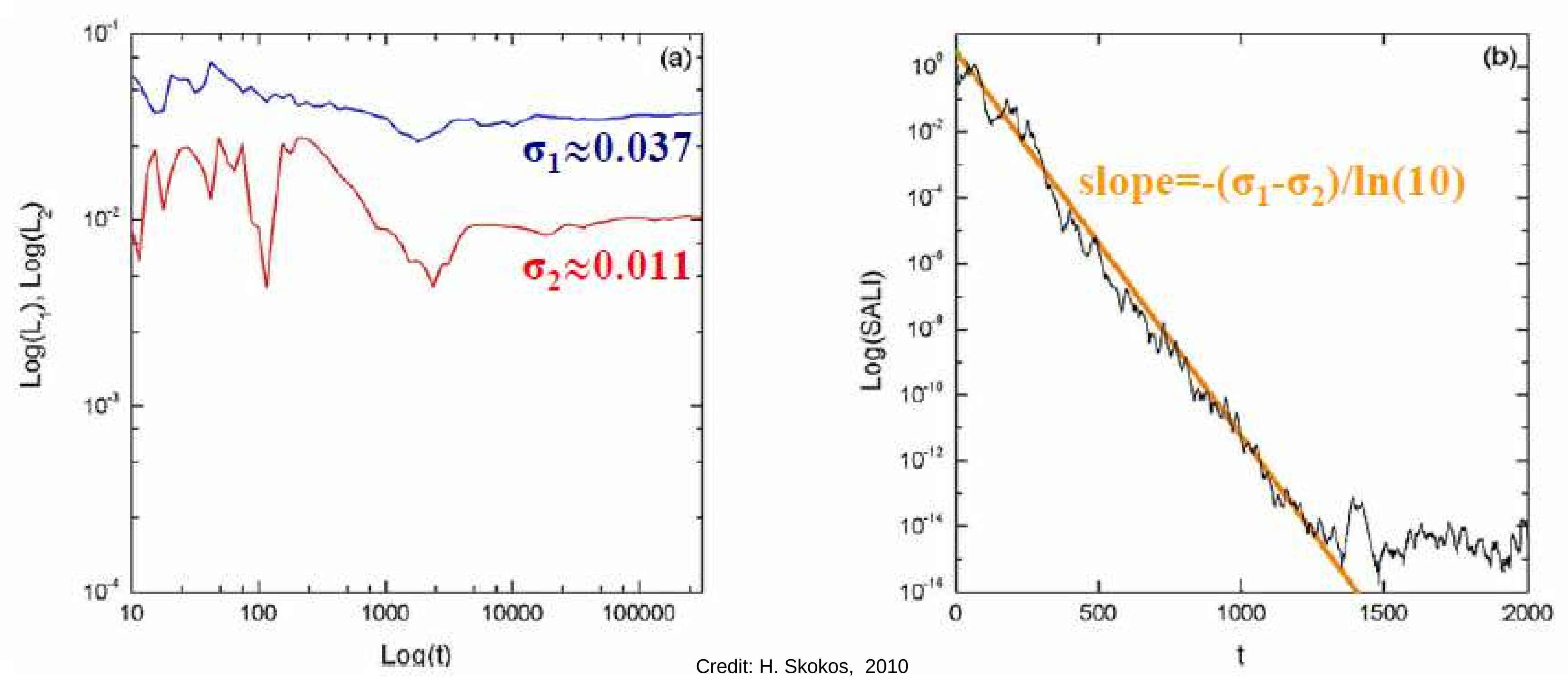
$$SALI(t) \propto e^{-(\chi_1-\chi_2)\cdot t}$$ χ_i … two largest LCEs

    - SALI $\rightarrow$ 0
  - Behavior for regular orbits:
    - SALI oscillates within the interval (0,2)

SALI / GALI - Examples

- Chaotic Motion of 3 D Hamiltonian $$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

$\omega_1=1$, $\omega_2=1.4142$, $\omega_3=1.7321$, $H=0.09$

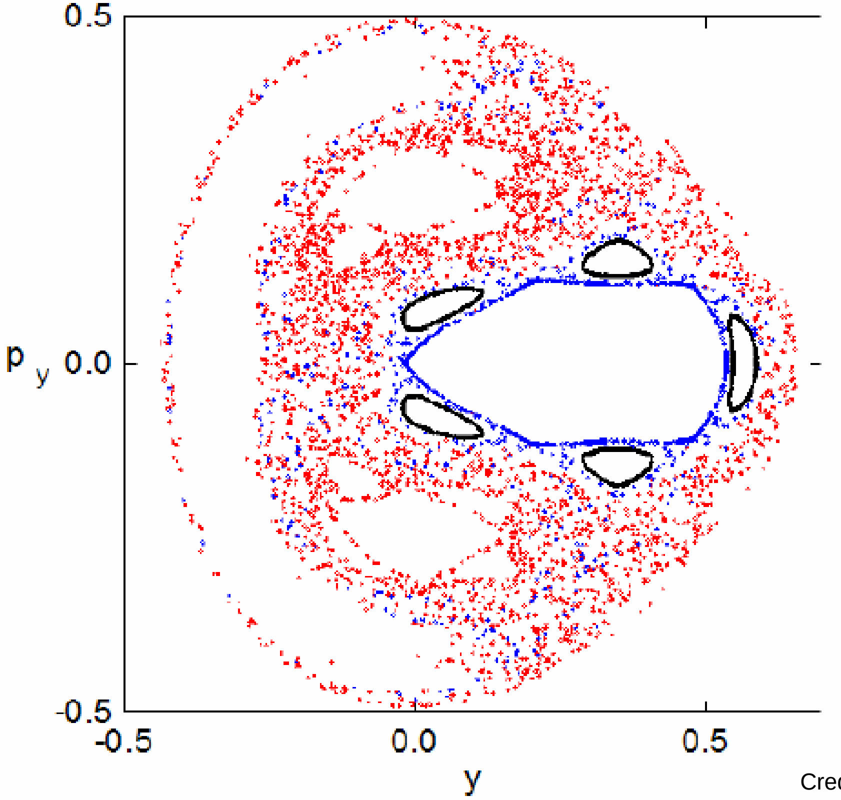


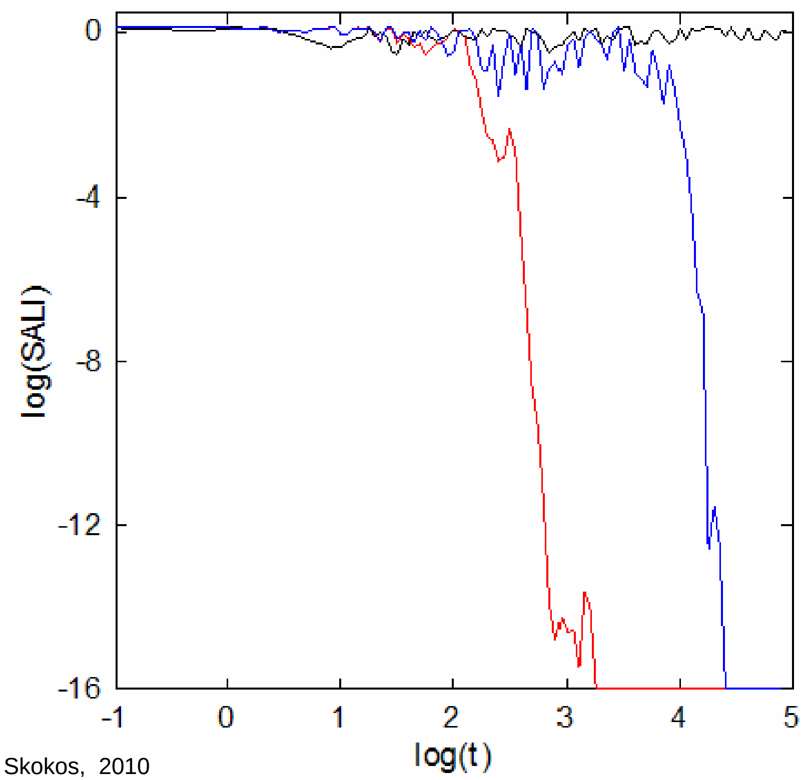
Credit: H. Skokos, 2010

SALI / GALI - Examples

- Application to Hénon-Heiles system:

For E=1/8 we consider the orbits with initial conditions:
Ordered orbit, x=0, y=0.55, p_x=0.2417, p_y=0
Chaotic orbit, x=0, y=-0.016, p_x=0.49974, p_y=0
Chaotic orbit, x=0, y=-0.01344, p_x=0.49982, p_y=0

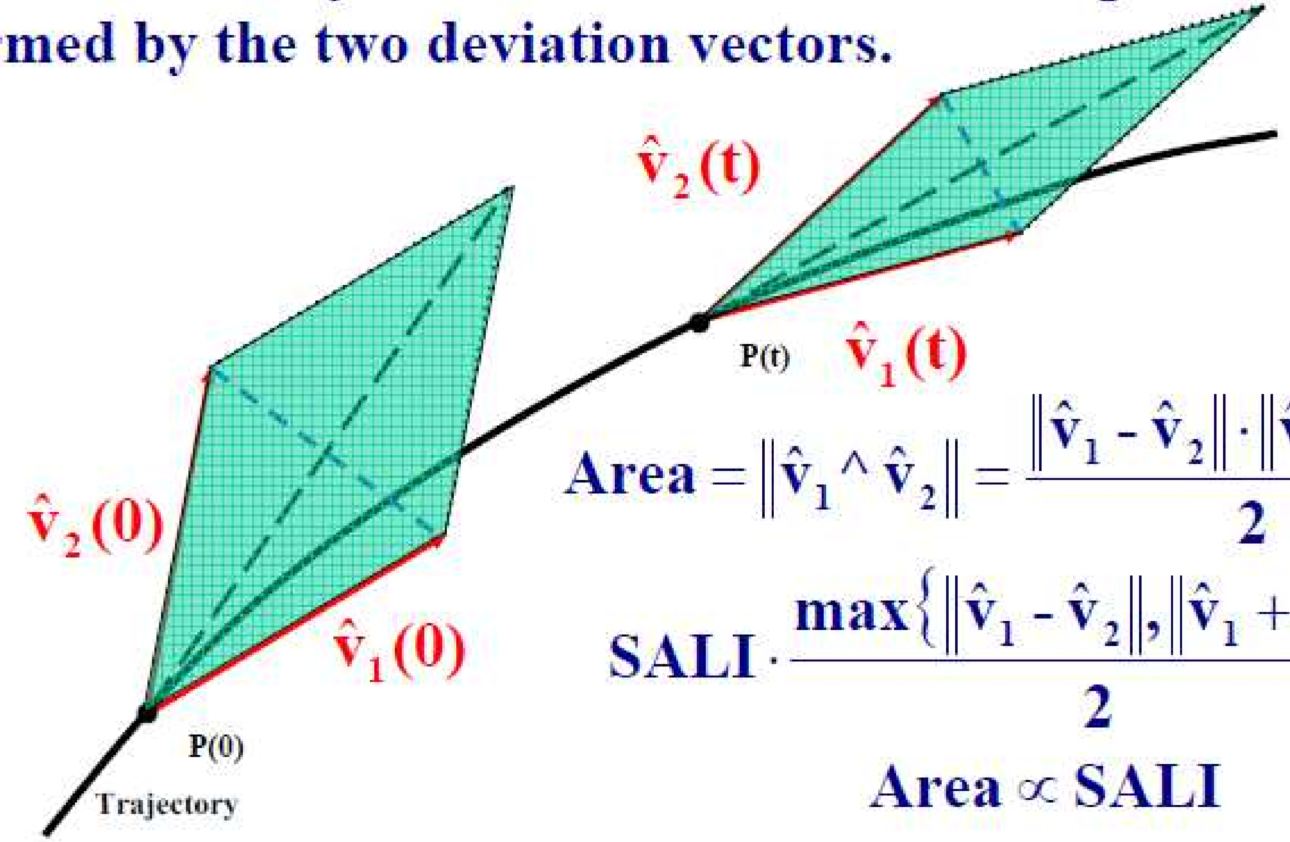




Credit: H. Skokos, 2010

SALI / GALI – Definition GALI

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.



$$\text{Area} = \left\|\hat{v}_1 \wedge \hat{v}_2\right\| = \frac{\left\|\hat{v}_1 - \hat{v}_2\right\| \cdot \left\|\hat{v}_1 + \hat{v}_2\right\|}{2} =$$

$$\text{SALI} \cdot \frac{\max\left\{\left\|\hat{v}_1 - \hat{v}_2\right\|, \left\|\hat{v}_1 + \hat{v}_2\right\|\right\}}{2} \Rightarrow$$

$$\text{Area} \propto \text{SALI}$$

Credit: H. Skokos, 2010

SALI / GALI – Definition GALI

- Definition GALI ("Generalized Alignment Index")
  - Orbit in n-dim space with initial condition:
  - k deviation vectors $2 \leq k \leq 2N$
  - Define GALI as:

$$\mathbf{GALI}_k(\mathbf{t}) = \left\| \hat{\mathbf{v}}_1(\mathbf{t}) \wedge \hat{\mathbf{v}}_2(\mathbf{t}) \wedge \ldots \wedge \hat{\mathbf{v}}_k(\mathbf{t}) \right\| \qquad \hat{\mathbf{v}}_1(\mathbf{t}) = \frac{\mathbf{v}_1(\mathbf{t})}{\left\| \mathbf{v}_1(\mathbf{t}) \right\|}$$

  - Wedge product:

$$\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \begin{vmatrix} \mathbf{v}_{1 i_1} & \mathbf{v}_{1 i_2} & \cdots & \mathbf{v}_{1 i_k} \\ \mathbf{v}_{2 i_1} & \mathbf{v}_{2 i_2} & \cdots & \mathbf{v}_{2 i_k} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{k i_1} & \mathbf{v}_{k i_2} & \cdots & \mathbf{v}_{k i_k} \end{vmatrix} \hat{\mathbf{e}}_{i_1} \wedge \hat{\mathbf{e}}_{i_2} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_k}$$

SALI / GALI

- Properties of GALI

  – Behavior for chaotic orbits:

$$GALI_k(t) \propto e^{-[(\chi_1-\chi_2)+(\chi_1-\chi_3)+\ldots+(\chi_1-\chi_k)]\cdot t}$$

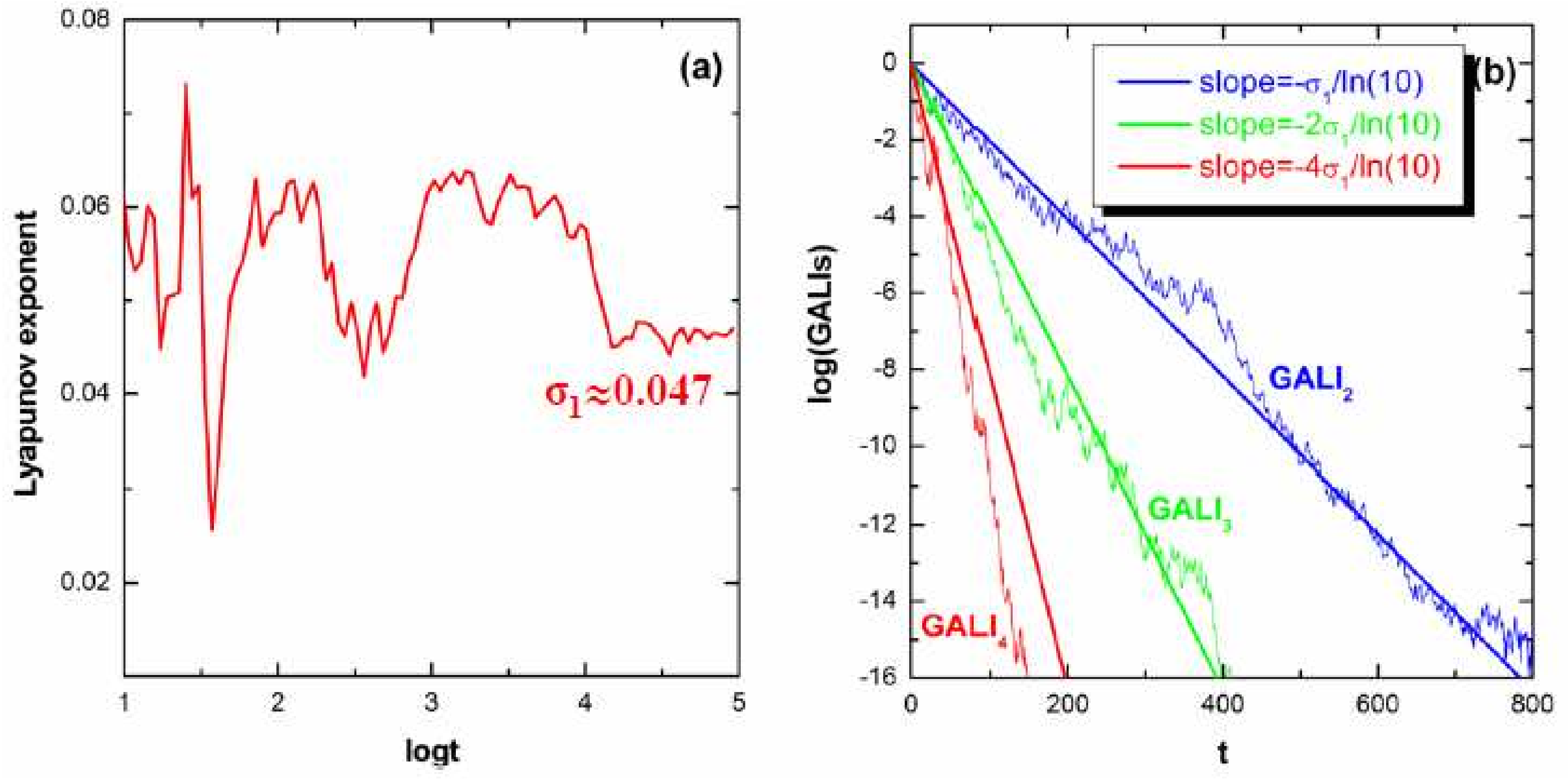
χ_i … k largest LCEs

  – Behavior for regular orbits:

$$GALI_k(t) \propto \begin{cases} constant \; if \; 2 \leq k \leq M \\ \frac{1}{t^{(k-M)}} \quad if \; M < k \leq 2N-M \\ \frac{1}{t^{2(k-N)}} \quad if \; 2N-M < k \leq 2N \end{cases}$$

SALI / GALI - Examples

- Behaviors of GALI for Chaotic Motion of 2 D Hamiltonian (Henon-Heiles system)



Credit: H. Skokos, 2010

Relative Lyapunov Indicator (RLI)

- RLI introduced by Z. Sandor, *et al*., 2004

- Definition of RLI:

  – LI difference for "base" orbit and its "shadow"

$$\Delta LI(\mathrm{x}_0; j) = \|LI(\mathrm{x}_0 + \mathbf{\Delta}\mathrm{x}; j) - LI(\mathrm{x}_0; j)\|$$

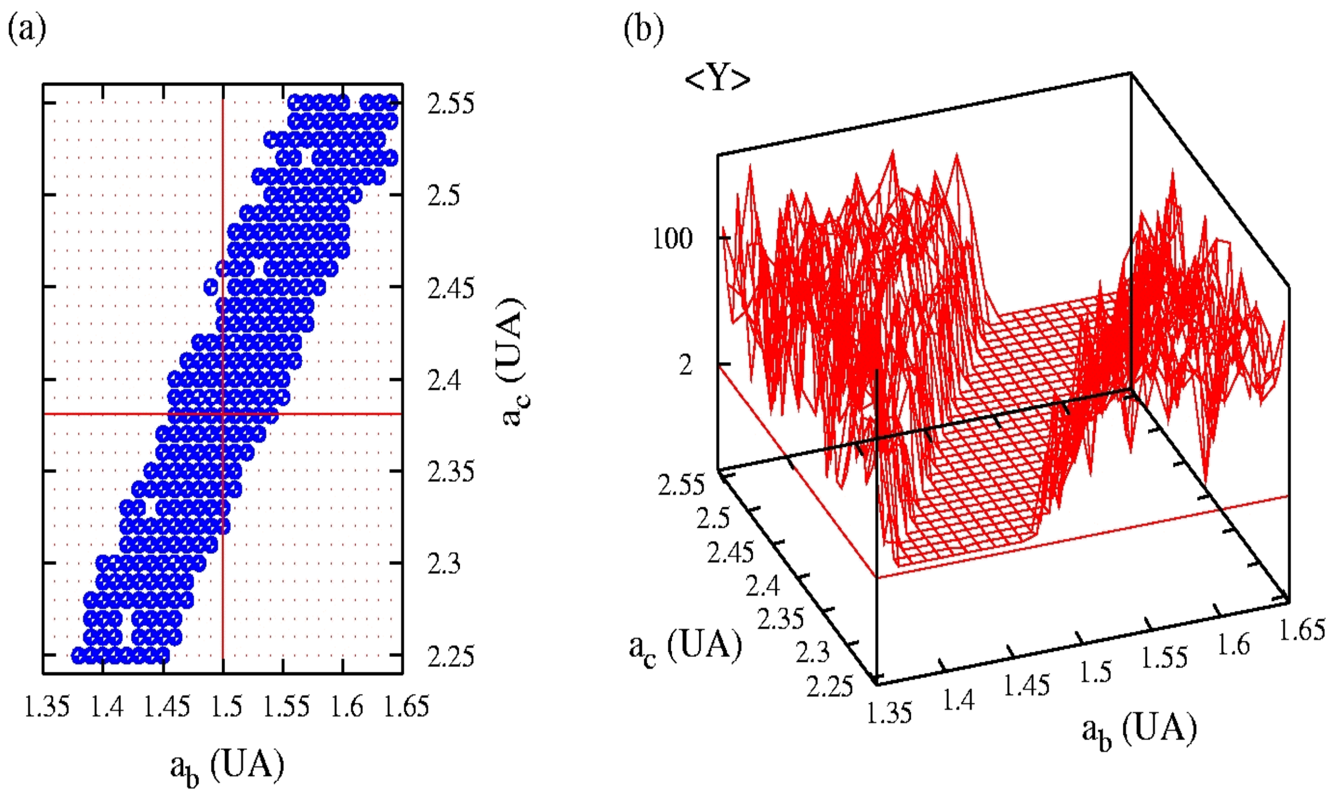
  – define RLI as

$$RLI(t) = \langle \Delta LI(\mathrm{x}_0) \rangle_t = \frac{1}{t} \sum_{i=1}^{t/\delta_t} \Delta LI(\mathrm{x}_0, i \cdot \delta_t)$$

- RLI values for chaotic motion are several orders of magnitude higher than for regular motion

Applications of Fast Chaos Indicators

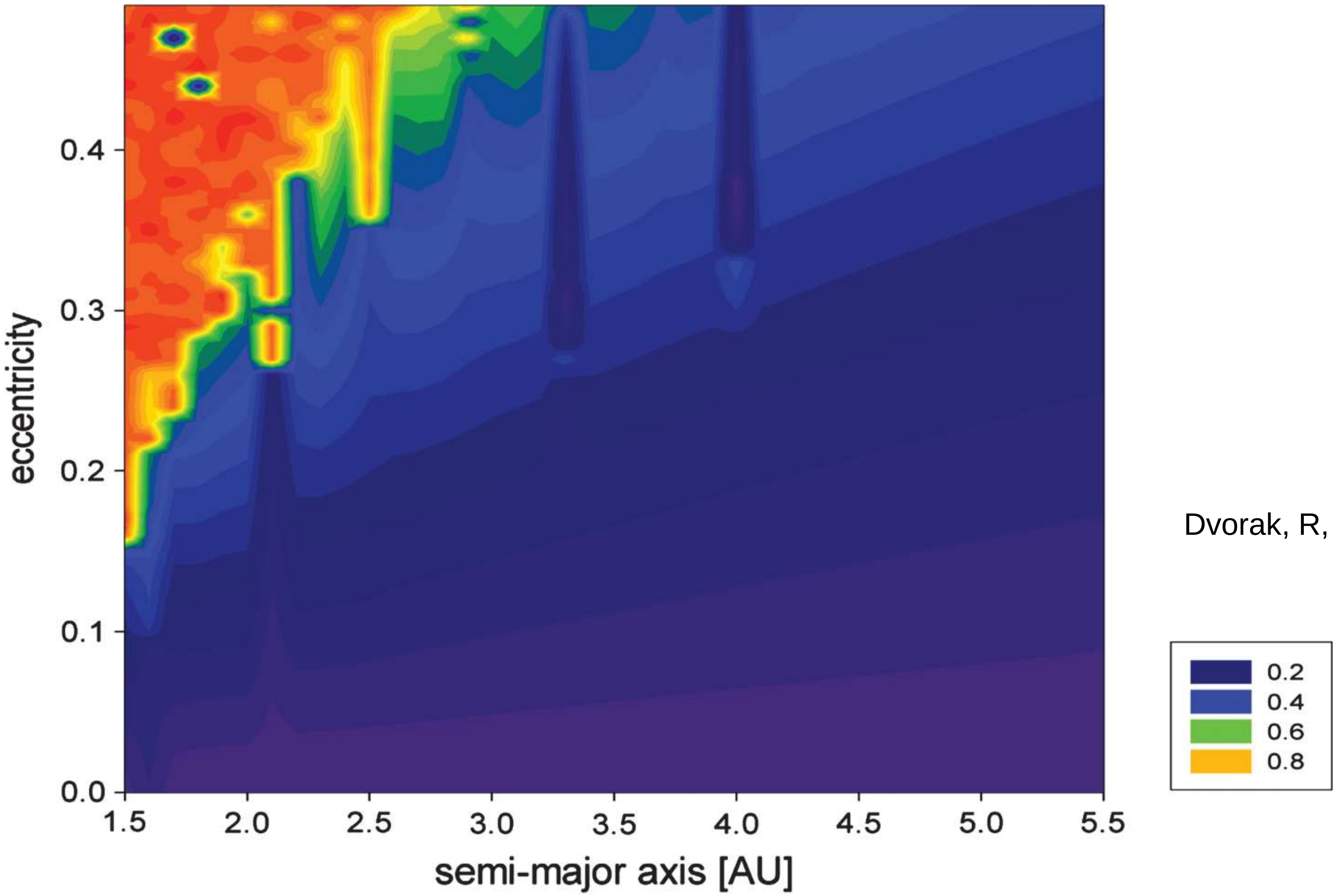
- Usage of MEGNO <Y> for HD 160692 system: 2:1 mean motion resonance



Bois, E., Kiseleva-Eggleton, L., Rambaux, N., Pilat-Lohinger, E., 2003, ApJ 598, 1312

Applications of Fast Chaos Indicators

- Stability map for Earth-like planet in the a_{Jup}, e_{Jup} for Sun-Jupiter system



Dvorak, R, *et al.*, Astrobiology, Vol. 10, No 1, 2010

Summary

- Classical methods to distinguish between regular and chaotic dynamical states like Lyapunov Characteristic Numbers (LCN), Poincare Surfaces of Section, require costly computations over long evolutionary times

- Fast Chaos Indicators (FLI, MEGNO, SALI/GALI, …) represent useful and efficient methods to distinguish between regular and chaotic planetary configurations

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