On the use of different coordinate systems in Celestial Mechanics

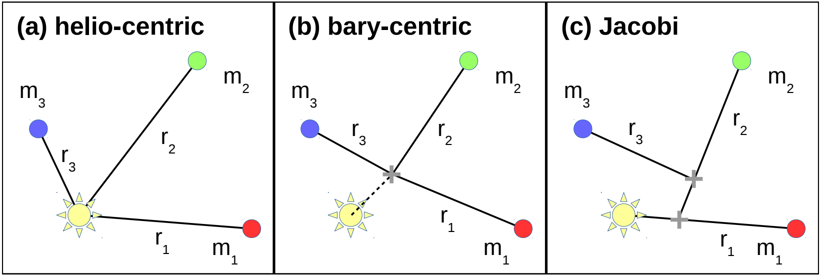
Ákos Bazsó

ADG Seminar, November 2016

Outline

Coordinate systems

(a) Heliocentric coordinates (HCO)
(b) Barycentric coordinates (BCO)
(c) Jacobi coordinates (JCO)
(d) Poincaré coordinates (PCO)



Typical usage cases

- **Heliocentric**
  - Solar system studies
  - Classical perturbation theory (Laplace-Lagrange theory)
- **Barycentric**
  - Extra-solar systems – radial velocity, astrometry
- **Jacobi**
  - (Hierarchical) 3-body problem
- **Poincaré**
  - Symplectic integration methods (MVS)

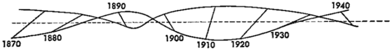


Figure 6.1 *The orbits of Sirius A and B*

from: Danby (1992)

Definitions

- $N + 1$ point masses m_i, $i = 0 \ldots N$
- Inertial system position/velocity vectors $(\mathbf{X}_i, \dot{\mathbf{X}}_i)$

Definitions

- $N + 1$ point masses m_i, $i = 0 \ldots N$
- Inertial system position/velocity vectors $(\mathbf{X}_i, \dot{\mathbf{X}}_i)$
- Generalized linear momentum: $\mathbf{p}_i = m_i \dot{\mathbf{X}}_i$
- Generalized coordinates: $\mathbf{q}_i = \mathbf{X}_i$

Definitions

- $N + 1$ point masses m_i, $i = 0 \dots N$
- Inertial system position/velocity vectors $(\mathbf{X}_i, \dot{\mathbf{X}}_i)$
- Generalized linear momentum: $\mathbf{p}_i = m_i \dot{\mathbf{X}}_i$
- Generalized coordinates: $\mathbf{q}_i = \mathbf{X}_i$
- Hamiltonian system $H(\mathbf{p}, \mathbf{q}, t)$
- Canonical variables $(\mathbf{p}, \mathbf{q})$:

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= -\frac{\partial H}{\partial \mathbf{q}} \\
\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} &= \frac{\partial H}{\partial \mathbf{p}}
\end{aligned}
$$

Barycentric Coordinates
a.k.a. Center-of-mass coord.

- Barycenter (center of mass) of system

$$\mathbf{X}_{BC} = \frac{1}{M}\sum_{n=0}^{N} m_n \mathbf{X}_n, \quad M = \sum_{n=0}^{N} m_n \; \dots \text{ total mass}$$

- Barycentric vectors — origin shifted to $\mathbf{X}_{BC}$

$$\mathbf{b}_i = \mathbf{X}_i - \mathbf{X}_{BC}$$

- Hamiltonian

$$H(\mathbf{p},\mathbf{q},t) = T(\mathbf{p}) + U(\mathbf{q}) = \\ = \left(\sum_{n=0}^{N} \frac{\mathbf{p}_n^2}{2m_n}\right) - G\left(\sum_{n=0}^{N}\sum_{k=n+1}^{N} \frac{m_n m_k}{\|\mathbf{q}_n - \mathbf{q}_k\|}\right)$$

Heliocentric Coordinates

or Astrocentric coord.

- Heliocentric vectors — origin shifted to $\mathbf{X}_0$

$$\mathbf{h}_i = \mathbf{X}_i - \mathbf{X}_0$$

- $(\mathbf{p}_i, \mathbf{q}_i) = (m_i\dot{\mathbf{h}}_i, \mathbf{h}_i)$ is not a canonical set of variables
- Hamiltonian $H = H_0 + H_1$
- H_0 = integrable 2-body part

$$H_0 = \sum_{n=1}^{N} \left(\frac{\mathbf{p}_n^2}{2m_n} - \frac{G(m_0 + m_n)m_n}{\|\mathbf{q}_n\|} \right)$$

- H_1 = small perturbation ($\mathcal{O}(m_i/m_0)$)

Jacobi Coordinates
from: Beaugé, Ferraz-Mello, Michtchenko (2008)

Jacobi canonical coordinates

$$\mathbf{j}_0 = \mathbf{X}_0$$

$$\mathbf{j}_1 = \mathbf{X}_1 - \mathbf{X}_0$$

$$\mathbf{j}_2 = \mathbf{X}_2 - \frac{1}{\sigma_1}(m_0\mathbf{X}_0 + m_1\mathbf{X}_1)$$

$$\mathbf{j}_3 = \mathbf{X}_3 - \frac{1}{\sigma_2}(m_0\mathbf{X}_0 + m_1\mathbf{X}_1 + +m_2\mathbf{X}_2)$$

$$\vdots$$

$$\mathbf{j}_i = \mathbf{X}_i - \frac{1}{\sigma_{i-1}}\sum_{n=0}^{i-1} m_n\mathbf{X}_n$$

$$\sigma_i = \sum_{n=0}^{i} m_n \ \ldots \text{ partial sum of masses}$$

Jacobi Coordinates

from: Beaugé, Ferraz-Mello, Michtchenko (2008)

- Hamiltonian $H = H_0 + H_1$
- H_0 = unperturbed part — m_n moving around "body" of mass σ_{n-1}

$$H_0 = \sum_{n=1}^{N} \left(\frac{\mathbf{p}_n^2}{2\rho_n} - \frac{G\sigma_n \rho_n}{\|\mathbf{q}_n\|} \right)$$

- H_1 = interaction part
- reduced masses $\rho_n = m_n \frac{\sigma_{n-1}}{\sigma_n}$

Poincaré Coordinates

a.k.a. Democratic-Heliocentric or Mixed-Variables coord.

Poincaré canonical coordinates

- heliocentric position vectors

$$\mathbf{q}_i = \mathbf{X}_i - \mathbf{X}_0$$

- barycentric velocity vectors

$$\mathbf{p}_i = \dot{\mathbf{X}}_i - \dot{\mathbf{X}}_{BC}$$

Poincaré Coordinates

a.k.a. Democratic-Heliocentric or Mixed-Variables coord.

Poincaré canonical coordinates

- Hamiltonian $H = H_0 + H_1$
- H_0 = unperturbed part

$$H_0 = \sum_{n=1}^{N} \left(\frac{\mathbf{p}_n^2}{2\beta_n} - \frac{G(m_0 + m_n)\beta_n}{\|\mathbf{q}_n\|} \right)$$

- H_1 = interaction part
- reduced masses $\beta_n = \frac{m_0 m_n}{m_0 + m_n}$

Definition of Orbital Elements

Definition (Stiefel & Scheifele, 1971)

An orbital element φ is a linear function of time t in the unperturbed case:

$$\varphi(t) = a + bt$$

Example

- semi-major axis $a(t) = \text{const.}$
- mean anomaly $M(t) = M_0 + n\,t$

Orbital elements

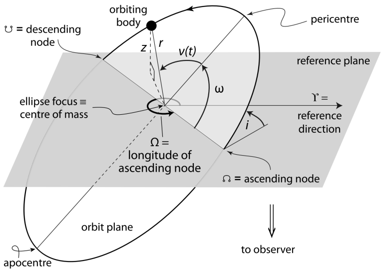
- Convert coordinates to orbital elements

$$(x, y, z, \dot{x}, \dot{y}, \dot{z}) \mapsto (a, e, i, \omega, \Omega, \nu)$$

- Perturbed case: orbital elements varying non-linearly with time
- (a, e) from position and velocity vectors

$$-\frac{GM}{2a} = \frac{\|\dot{\mathbf{X}}\|^2}{2} - \frac{GM}{\|\mathbf{X}\|}$$

$$e = \sqrt{1 - \frac{\|\mathbf{L}\|^2}{GMa}}$$



from: Perryman (2011)

Generalized Orbital Elements

- Heliocentric

$$H_{0,n} = \frac{\mathbf{p_H}_n^2}{2m_n} - \frac{G(m_0 + m_n)m_n}{\|\mathbf{q_H}_n\|}$$

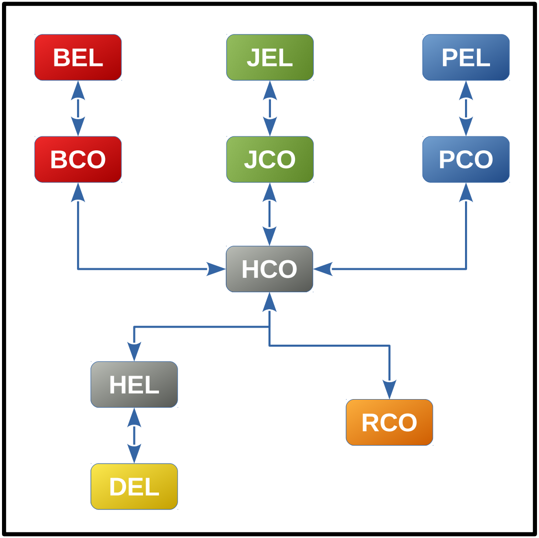
- Jacobi

$$H_{0,n} = \frac{\mathbf{p_J}_n^2}{2\rho_n} - \frac{G\sigma_n\rho_n}{\|\mathbf{q_J}_n\|}$$

- Poincaré

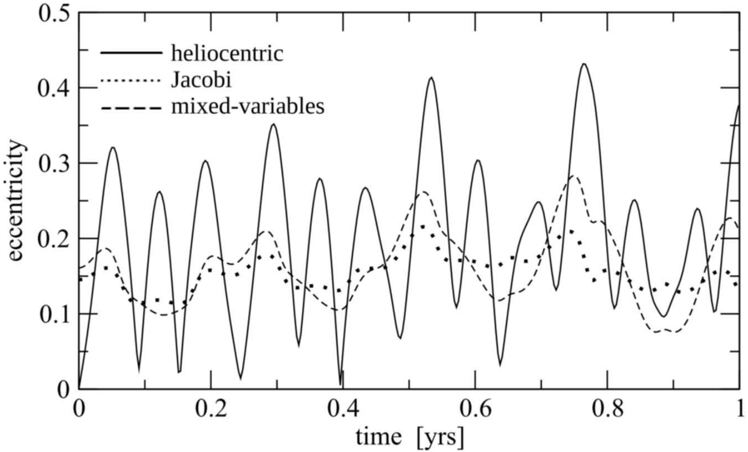
$$H_{0,n} = \frac{\mathbf{p_P}_n^2}{2\beta_n} - \frac{G(m_0 + m_n)\beta_n}{\|\mathbf{q_P}_n\|}$$

Coordinate conversions



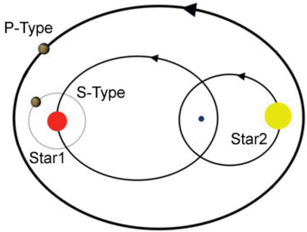
Example

from: Beaugé, Ferraz-Mello, Michtchenko (2008), Fig. 1.8



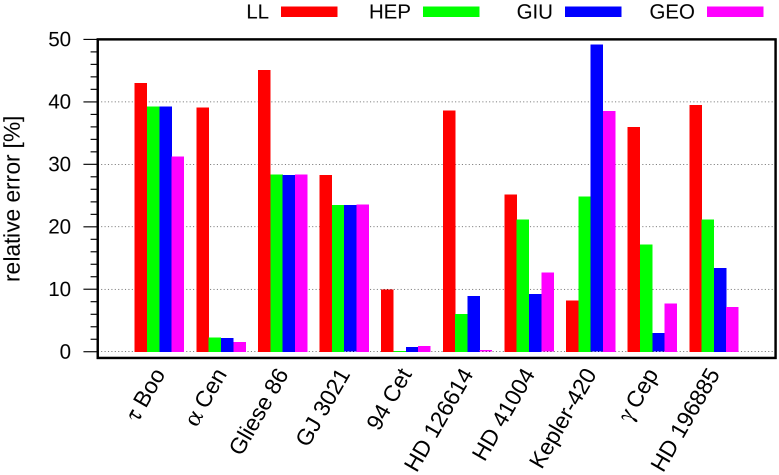
Application

- Binary star system with S-type extrasolar planet
- Apsidal precession frequency $g \sim \mathrm{d}\omega/\mathrm{d}t$
- Determine $g = g(a_P, a_B, e_B)$ from analytical perturbation theory
  - Laplace-Lagrange — LL
  - Heppenheimer (1978) — HEP
  - Georgakarakos (2003) — GEO
  - Giuppone et al. (2011) — GIU



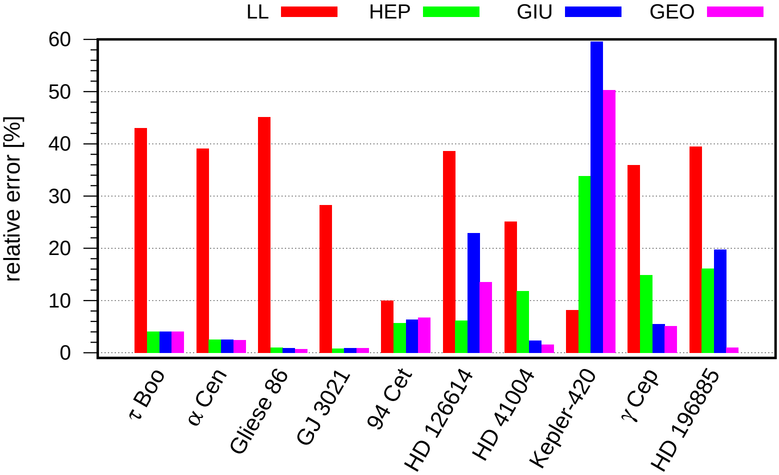
by: R. Schwarz

Results



using heliocentric coordinates

Results



using Jacobi coordinates

Summary

- 4 main astrodynamical coordinate systems:
  BCO, HCO, JCO, PCO
- Generalized orbital elements associated to each system:
  BEL, HEL, JEL, PEL
- Coordinate conversions: software library **`libcoocvt`**