SPH for simulating impacts and collisions

Thomas I. Maindl
University of Vienna

Jan 22, 2013

Christoph Schäfer, Roland Speith (Eberhard Karls University of Tübingen)

Agenda

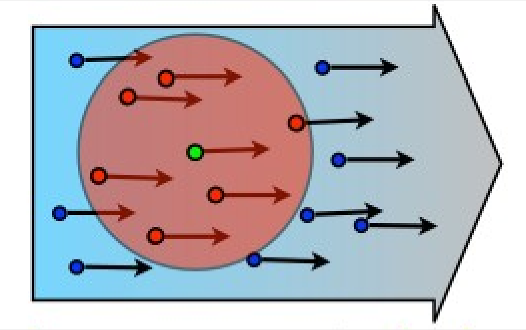
- SPH quick overview
- Solid body physics overview
- Tests and first results
- Outlook
- References

Smoothed Particle Hydrodynamics (SPH)

- SPH origin: simulating hydrodynamic problems in astrophysics
  - Extended to elasto-plastic dynamics & self gravity
- Application examples include
  - Cosmology
  - Star and planet formation
  - Interactions of stars, black holes,...
  - Accretion discs
  - Material science
  - …

SPH is a mesh-free Lagrangian particle method

- Completely different from finite difference and finite volume methods → well suited for comparisons
- The simulated system is represented as a set of interacting “SPH particles” which
  - carry all physical properties of their “fluid part”
  - determine the density in their region (= number of SPH particles in a specific region)
  - move like point masses governed by the Lagrangian form of the equations of motion
  - are to be interpreted as a numerical vehicle rater than physical particles



Speith (2012)

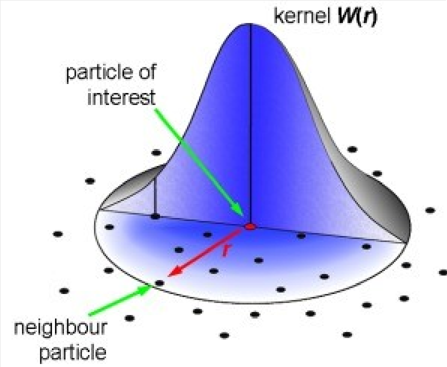
SPH principle

- System of coupled PDEs → system of ODEs

1. Smooth quantities via kernel convolution

$$f(\boldsymbol{r}) \longrightarrow \int f(\boldsymbol{r}')\, W(|\boldsymbol{r}-\boldsymbol{r}'|)\, \mathrm{d}\boldsymbol{V}' = f(\boldsymbol{r}) + \mathcal{O}(h^2)$$

h... smoothing length, "radius" of kernel, determines spatial resolution



NUI Galway (2012)

2. Remove spatial derivatives

$$\nabla f(\boldsymbol{r}) \longrightarrow \int f(\boldsymbol{r}')\, \nabla W(|\boldsymbol{r}-\boldsymbol{r}'|)\, \mathrm{d}\boldsymbol{V}'$$

3. Discretize

$$\nabla f(\boldsymbol{r}^i) \approx \sum_j \frac{m^j}{\rho^j}\, f(\boldsymbol{r}^j)\, \nabla W(|\boldsymbol{r}^i-\boldsymbol{r}^j|, h)$$

Example: equivalent formulations

- Equivalent to $\nabla f(\boldsymbol{r^i}) \approx \sum_j \frac{m^j}{\rho^j} f(\boldsymbol{r^j})\, \nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h)$

  to $\mathcal{O}(h^2)$ ensuring that derivatives of constant functions vanish:

$$
\begin{aligned}
\nabla f^i &= \frac{\rho^i}{\rho^i} \nabla f^i = \frac{1}{\rho^i} \left[\nabla(\rho^i f^i) - f^i \nabla \rho^i \right] \\
&\approx \frac{1}{\rho^i} \sum_j \frac{m^j}{\rho^j} \rho^j f^j \nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h) - \frac{f^i}{\rho^i} \sum_j \frac{m^j}{\rho^j} \rho^j \nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h) \\
&= \frac{1}{\rho^i} \sum_j m^j (f^j - f^i) \nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h)
\end{aligned}
$$

Problem-dependent equation formulation

- Example Euler equation

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p \qquad \frac{\mathrm{d}\boldsymbol{v}^i}{\mathrm{d}t} = -\sum_j \frac{m^j}{\rho^j}\frac{p^j}{\rho^i}\,\nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h)$$

- Equivalent to $\mathcal{O}(h^2)$ and numerically more stable:

$$\frac{\mathrm{d}\boldsymbol{v}^i}{\mathrm{d}t} = -\sum_j m^j \frac{p^i + p^j}{\rho^i \rho^j}\,\nabla W(|\boldsymbol{r^i} - \boldsymbol{r^j}|, h)$$

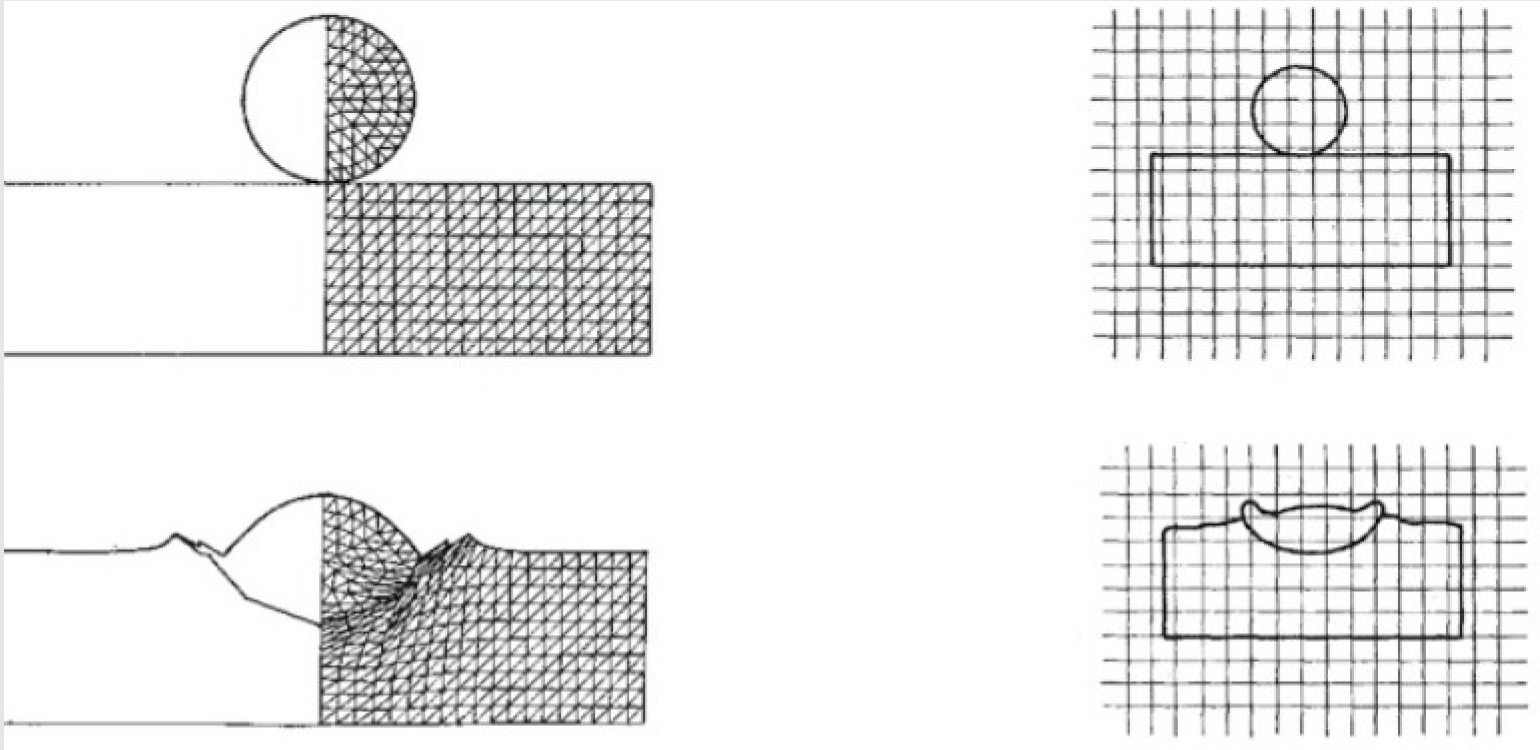
- using λ=1 in the identity

$$\frac{1}{\rho^{2-\lambda}}\nabla\frac{p}{\rho^{\lambda-1}} = \frac{1}{\rho}\nabla p - \frac{p}{\rho^{\lambda}}\nabla\rho^{\lambda-1}$$

SPH for impact simulations

Mesh-free Lagrangian method provides natural reference frame for treating deformations and fragmentation

Lagrange scheme | Euler scheme

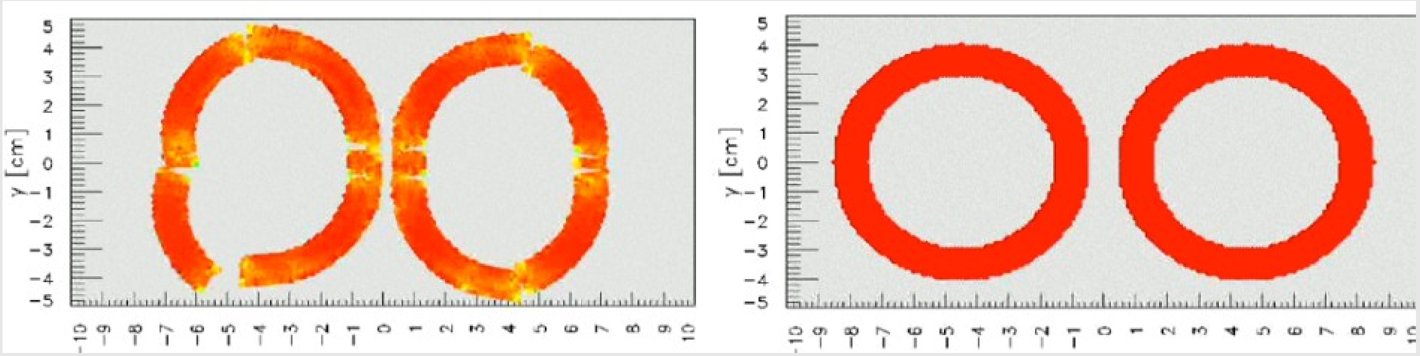


Lagrange scheme: number of “grid points” resolving the object is not reduced by deformation

Speith (2012)

Special SPH topics

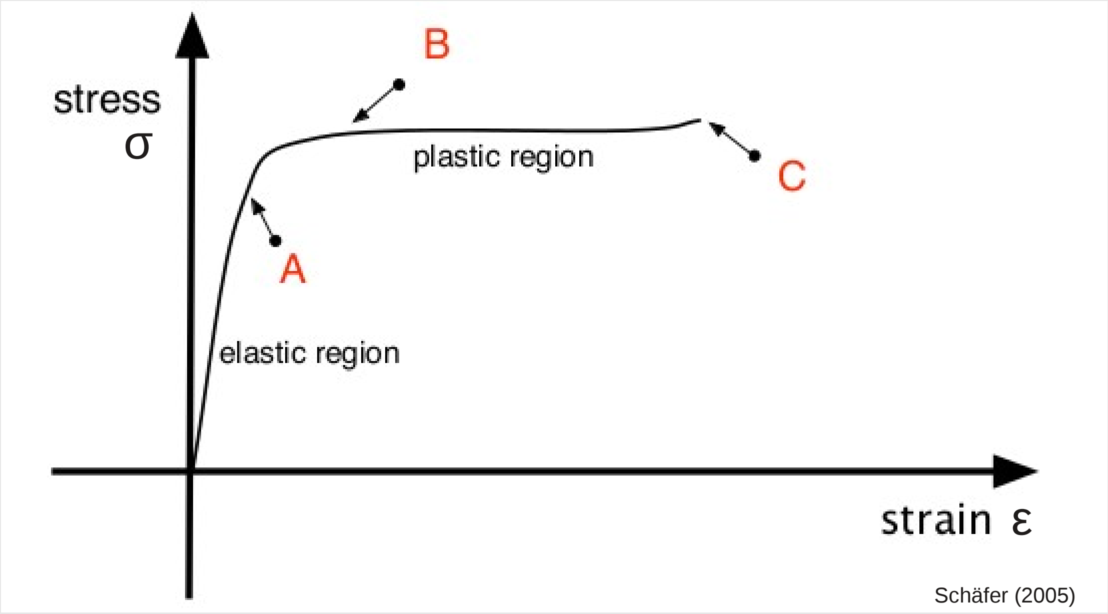
- Tensile instability
  - Artificial clumping leading to unphysical results



Speith (2012)

  - Solution: small artificial repulsive stress (Monaghan 2000)
- There are other SPH issues
  - XSPH
    - Smooth velocities preventing mutual penetration
  - Artificial viscosity
    - Prevents mutual penetration of particles
  - Integrate density rather than using ρ = Σ m W for stable surfaces
  - …

Solid bodies: continuum mechanics



A) Hooke's law: elasticity, deviatoric stress rate proportional to strain rate

B) Yielding relations: plasticity by modifying stresses beyond the elastic limit

C) Damage model and brittle failure for tensile stresses beyond material strength

Material equations

Continuity equation (mass conservation):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\frac{\partial v^\alpha}{\partial x^\alpha}$$

EOM (conservation of momentum):

$$\frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = \frac{1}{\rho}\,\frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} - \frac{\partial \Phi}{\partial x^\alpha}, \qquad \text{stress tensor } \sigma^{\alpha\beta} = -p\,\delta^{\alpha\beta} + S^{\alpha\beta}$$

Energy conservation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{p}{\rho}\,\frac{\partial v^\alpha}{\partial x^\alpha} + \frac{1}{\rho}\,S^{\alpha\beta}\dot{\varepsilon}^{\alpha\beta}, \qquad \text{strain rate tensor } \dot{\varepsilon}^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right)$$

Constitutive equation:

$$\frac{\mathrm{d}S^{\alpha\beta}}{\mathrm{d}t} = 2\mu\left(\dot{\varepsilon}^{\alpha\beta} - \frac{1}{3}\,\delta^{\alpha\beta}\dot{\varepsilon}^{\gamma\gamma}\right) + S^{\alpha\gamma}R^{\gamma\beta} + S^{\beta\gamma}R^{\gamma\alpha}$$

with rotation rate tensor $R^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} - \frac{\partial v^\beta}{\partial x^\alpha}\right)$

EOS: $\quad p = p(\rho, u)$

Plastic behavior: von Mises yielding criterion

Limit deviatoric stress tensor by

$$S^{\alpha\beta} = f\, S^{\alpha\beta}$$

with

$$f = \min\left[\frac{Y_0^2}{3\, J_2}, 1\right], \qquad J_2 = \frac{1}{2}\, S^{\alpha\beta} S^{\alpha\beta}$$

and the material dependent yield stress Y_0

Equation of state

- Connects the thermodynamic variables *ρ*, *p*, and *u* to close the set of equations
- Several analytical and semi-empirical approaches exist, e.g.,
  - Murnaghan EOS (isothermal only)

$$p = \frac{K_0}{n}\left[\left(\frac{\rho}{\rho_0}\right)^n - 1\right]$$

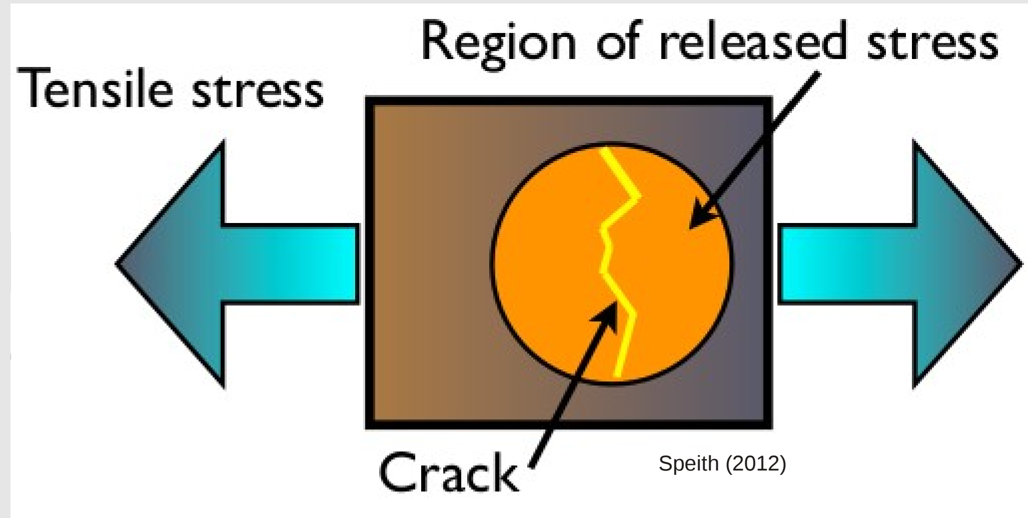
  - Tillotson (1962) EOS

$$p = \left(a + \frac{b}{\frac{u}{u_0\eta^2} + 1}\right)\rho u + A\mu + B\mu^2,\ \eta = \frac{\rho}{\rho_0},\ \mu = \eta - 1 \qquad (u < u_{iv})$$

$$p = a\rho u + \left[\frac{b\rho u}{\frac{u}{u_0\eta^2} + 1} + A\mu e^{-\beta\left(\frac{\rho_0}{\rho} - 1\right)}\right] e^{-\alpha\left(\frac{\rho_0}{\rho} - 1\right)} \qquad (u > u_{cv})$$

  - ANEOS EOS
    - Semi-analytical, not freely available

Fracture model



- Large enough local **strain** causes **flaws** in the solids to develop into **cracks**
- Cracks grow at half the speed of sound until the local stress is relieved
- Grady & Kipp (1980) damage model:
  - damage D with $0 \leq D \leq 1$
  - Stress ~ (1 – D)
- Modified stress tensor:

$$\sigma_{\alpha\beta}^{\text{damaged}} = \begin{cases} -p\delta_{\alpha\beta} + (1-D)S_{\alpha\beta} & , \; p \geq 0 \quad \text{(compression)} \\ -(1-D)p\delta_{\alpha\beta} + (1-D)S_{\alpha\beta} & , \; p < 0 \quad \text{(tension)} \end{cases}$$

Flaw distribution

- The Grady-Kipp damage model assumes a probability distribution of flaws
- Weibull (1939) distribution:

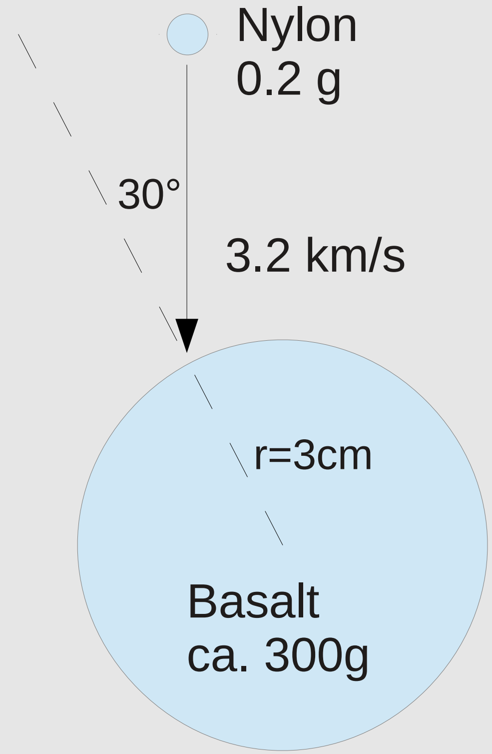
$$n(\varepsilon) = k\,\varepsilon^{m}$$

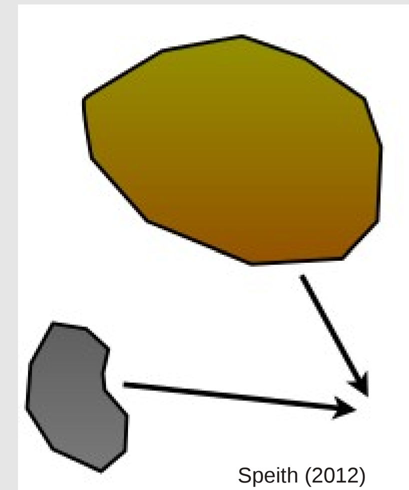
n … number of flaws per unit volume with activation thresholds $< \varepsilon$

k, m … material parameters

- Distribution parameters not easily measurable...

Numerical tests





First results

- Collisions of brittle bodies (basalt)
- Tillotson EOS, measured Weibull distribution parameters
- Projectile:
  - Spherical, e.g. small asteroid, 50cm radius
  - No flaws
- Target:
  - Spheroidal, e.g., irregular shaped small asteroid
    - Semi axes: 5m, 10m
- Impact velocity 1 km/s

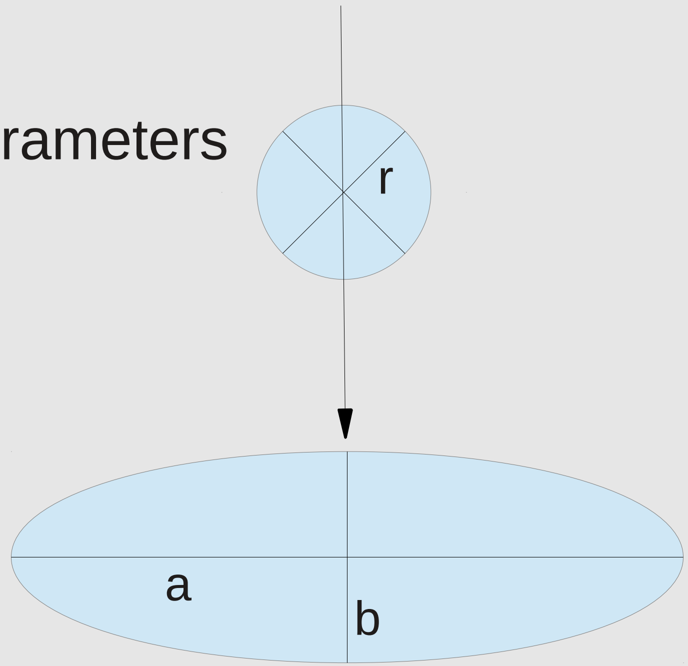


Table 2 (I) Material constants, cf. Benz & Asphaug (1999), (II) Weibull distribution parameters, cf. Nakamura et al. (2007) and Lange et al. (1984), respectively

	(I)		(II)	
	μ (GPa)	Y (GPa)	m	k (m^{-3})
Basalt	22.7	3.5	16	10^{61}
Ice	2.8	1	9.1	10^{46}

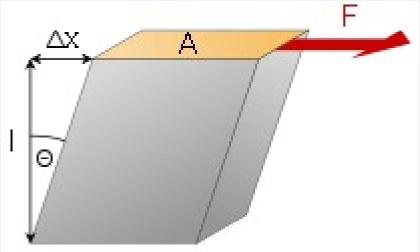
Tillotson EOS parameters and vaporization energy levels adopted from Benz & Asphaug (1999)

	ρ_0 (kg/m^3)	A (GPa)	B (GPa)	E_0 (MJ/kg)	E_{iv} (MJ/kg)	E_{cv} (MJ/kg)	a	b	α	β
Basalt	2700	26.7	26.7	487	4.72	18.2	0.5	1.50	5.0	5.0
Ice	917	9.47	9.47	10	0.773	3.04	0.3	0.1	10.0	5.0

Maindl et al. (2013)

Challenge: material constants

- Material constants
  - Shear modulus μ



  -
  - bulk modulus K, yield stress Y
  - Weibull distribution parameters
  - EOS coefficients

Table 2 (I) Material constants, cf. Benz & Asphaug (1999), (II) Weibull distribution parameters, cf. Nakamura et al. (2007) and Lange et al. (1984), respectively

	(I) μ (GPa)	Y (GPa)	(II) m	k (m^{-3})
Basalt	22.7	3.5	16	10^{61}
Ice	2.8	1	9.1	10^{46}

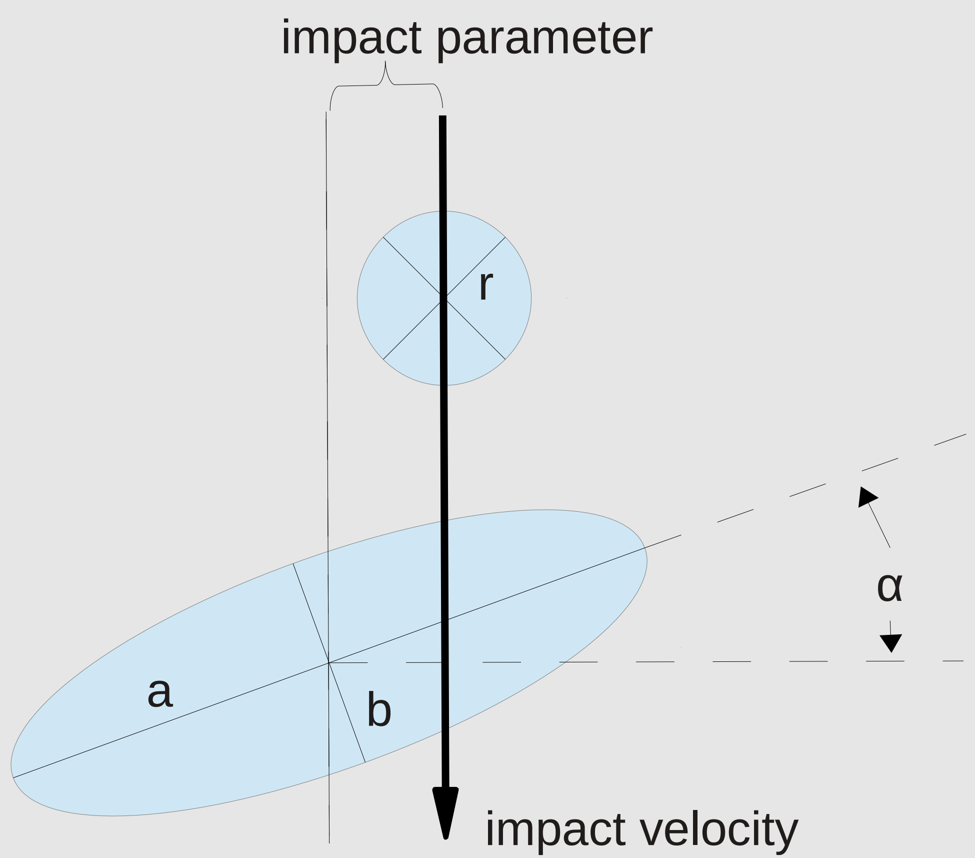
Tillotson EOS parameters and vaporization energy levels adopted from Benz & Asphaug (1999)

	ρ_0 (kg/m^3)	A (GPa)	B (GPa)	E_0 (MJ/kg)	E_{iv} (MJ/kg)	E_{cv} (MJ/kg)	a	b	α	β
Basalt	2700	26.7	26.7	487	4.72	18.2	0.5	1.50	5.0	5.0
Ice	917	9.47	9.47	10	0.773	3.04	0.3	0.1	10.0	5.0

Maindl et al. (2013)

Outlook

- Measure fragmentation and merging
- Goal: water in early planetary systems
  - What influence does water content have?
    - Fragmentation
    - Merging
  - Water in/on protoplanets
  - Influence of different water/ice distributions
  - Porous bodies
  - Self gravitation?
  - Link to n-body



References

Benz, W., Asphaug, E.: 1994, Icar 107, 98.

Benz, W., Asphaug, E.: 1999, Icar 142, 5.

Grady, D. E., Kipp, M. E.: 1980. Continuum modelling of explosive fracture in oil shale. Int. J. Rock Mech. Min. Sci. Geomech. Abstr. 17, 147.

Lange, M.A., Ahrens, T.J., Boslough, M.B.: 1984, Icar 58, 383.

Maindl, T.I., Schäfer, C., Speith, R., Süli, Á., Forgacs-Dajka, E., Dvorak, R.: 2013, SPH-based simulation of icy asteroid collisions. Astron. Nachrichten, in preparation.

Melosh, H.J.: 1989, Impact Cratering – A Geologic Process, Oxford Univ. Press, New York.

Monaghan, J.J.: 2000, SPH without a tensile instability. Journal of Computational Physics 159, 290.

Nakamura, A.M., Michel, P., Setoh, M.: 2007, JGR 112, E02001, doi:10.1029/2006JE002757.

NUI Galway: 2012, National University of Ireland, Galway, http://www.nuigalway.ie/mechbio/research/meshfree.html, accessed Sep 1, 2012.

Schäfer, C.: 2005, Application of Smooth Particle Hydrodynamics to selected Aspects of Planet Formation, Dissertation, Eberhard-Karls-Universität Tübingen.

Speith, R.: 2012, Smoothed Particle Hydrodynamics, Workshop at University of Vienna, Aug 1-3, 2012, unpublished.

Tillotson, J. H.: 1962. Metallic equations of state for hypervelocity impact. General Atomic Report GA-3216.

Weibull, W. A.: 1939. A statistical theory of the strength of materials [translated]. Ingvetensk. Akad. Handl. 151, 5.

Thank you!