

The Role of Resonances in Astrodynamical Systems

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Abstract

This review intends to highlight the importance of resonances of the orbits of planets and asteroids in our Solar System. Besides the well known 5:2 mean motion resonance between Jupiter and Saturn, we discuss how secular resonances act, where the motions of the longitudes of the perihelia and/or the longitudes of the nodes are involved. Resonances are especially important for the motions of small bodies in the Solar System from the Earth crossing Near Earth Asteroids to the far moving asteroids and comets in the Kuiper belt outside Pluto's orbit. Briefly we also discuss retrograde orbits in resonances which turn out to be much more stable than prograde ones. The presence and importance of resonances in the dynamics of extrasolar planetary systems is a new exciting field for celestial mechanics.

1 Introduction

In physics it is well known, that resonances play a fundamental role in any dynamical system. But, how important are resonances in astrodynamical systems? In this review we will deal first of all with our very own Solar System, where planets are trapped in so-called mean motion resonances causing chaos among asteroids and where celestial bodies generally suffer from strong perturbations due to secular resonances. With the discovery of extrasolar systems hosting planets which seem to be on highly eccentric orbits, that still are stable, another exiting chapter has been added to this topic. Such a behavior is understandable only, if these planets are locked in certain resonances. The examples show how diverse resonances can be: on the one hand they cause instability, on the other hand they seem to act as protective mechanisms.

We begin with a short description of the different kinds of orbital resonances which act in the motion of planets and small bodies in the planetary system which are Mean Motion Resonances (=MMR), Secular Resonances (=SR), Kozai Resonances (=KR) and Three Body Resonances (=TBR). Then two examples of motions of planets in resonances are discussed shortly: Jupiter and Saturn are close to a 5:2 MMR and Venus and Earth to the 13:8 MMR. Resonant effects are of special importance for all groups of asteroids, for the Near Earth Asteroids, the main belt asteroids, the Trojans and also the Kuiper belt objects moving outside the orbit of Neptune. We add a short chapter on retrograde orbits which are commonly more stable than prograde ones and finally, in an 'epilog', we briefly discuss the disputed large eccentricities of the orbits of extrasolar planets which may be connected to MMR between the planets in such systems.

Table 1: Maximum and Minimum values of the orbital elements a , e and i for 10^9 years, according to numerical integrations [9]

planet	a_{min}	a_{max}	e_{min}	e_{max}	i_{min}	i_{max}
Mercury	0.38710	0.38710	0.07874	0.29988	0.17600	11.72747
Venus	0.72332	0.72336	0.00002	0.07709	0.00076	4.91515
Earth	0.99997	1.00004	0.00002	0.06753	0.00075	4.49496
Mars	1.52354	1.52386	0.00008	0.13110	0.00291	8.60320
Jupiter	5.20122	5.20504	0.02513	0.06191	1.09172	2.06598
Saturn	9.51281	9.59281	0.00742	0.08959	0.55867	2.60187
Uranus	19.09807	19.33511	0.00008	0.07835	0.42170	2.73888
Neptune	29.91013	30.32452	0.00001	0.02317	0.77977	2.38597

2 Basic considerations

2.1 The orbital elements

An unperturbed motion of a celestial body in the gravitation field of the Sun can be described by 6 orbital elements which are constant in the case of no other forces acting¹. These elements are the 3 action like variables semimajor axis a , the eccentricity of the ellipse e and inclination i of the orbital plane with respect to the reference plane (the orbital plane of the Earth, see Figures 1 and 2). The other elements are the angles: perihelion² distance to the ascending node ω , the longitude of the ascending node Ω ³ and the true anomaly v . The mean motion n is connected to the semimajor axis via Kepler's 3rd law $n^2 a^3 = const.$ The mean longitude $\lambda = \Omega + \omega + M$ is a fictitious angle measured in two different planes and is commonly used in celestial mechanics; $M = n \cdot t$ is the mean anomalie. In the 0th order approximation one ignores the perturbations of the other planets and assigns fixed values of the elements to each planet, which are true for a certain epoch. To give an idea of the smallness of the variations of a , e and i when the mutual perturbations between the planets are acting we list the maximum and minimum values for these orbital elements in Table 1.

2.2 Mean Motion Resonances

To describe the motion of planets, respectively how they perturb reciprocally, it is convenient when to use the orbital elements introduced by the French astronomer C.E. Delaunay (1816-1872) for $i = 1, \dots, 8$ involved planets:

$$\begin{aligned}
 L_i &= \kappa_i \sqrt{a} \\
 G_i &= L_i \sqrt{(1 - e_i^2)} \\
 H_i &= G_i \cos i_i \\
 l_i &= M_i \\
 g_i &= \omega_i \\
 h_i &= \Omega_i
 \end{aligned} \tag{1}$$

¹According to Kepler's 1st law the motion of a planet is an ellipse.

²The perihelion is the minimum distance between the planet and the Sun.

³measured from the vernal equinox which is the intersection between the equator of the Earth and the ecliptic plane.

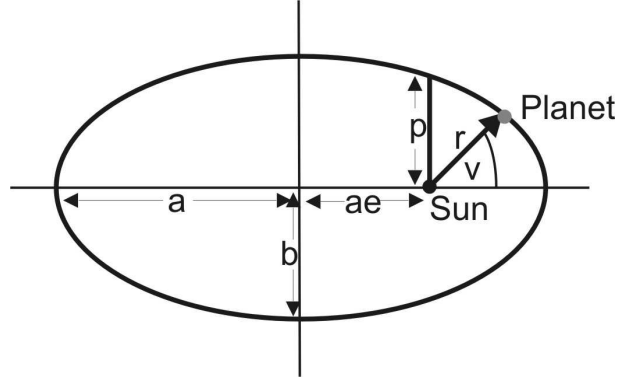


Figure 1: A Keplerian ellipse: v is the true anomaly, r is the position vector, a and b are the semimajor axes, $p = a(1 - e^2)$ the 'parameter' and e is the eccentricity.

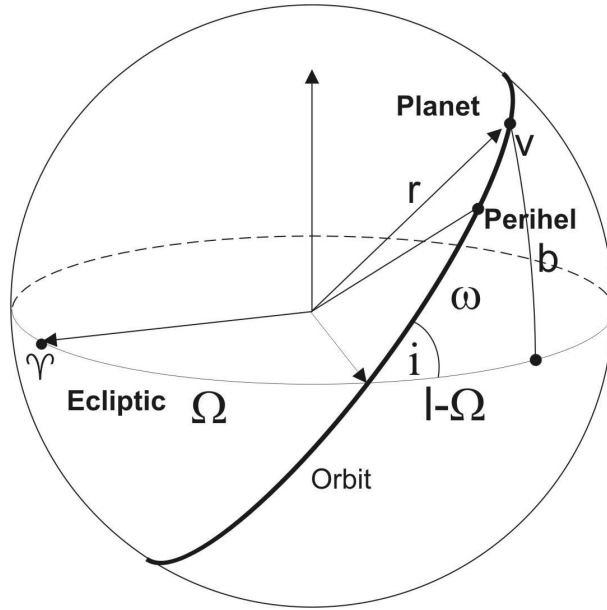


Figure 2: The Kepler elements in space referring to the vernal equinox γ : l and b are the ecliptic longitude respectively the ecliptic latitude, i is the inclination between the orbital plane of the planet and the orbital plane of the Earth, the ecliptic. Ω and ω refer to the longitude of the ascending node and the perihelion distance from the latter one.

In the former equations we used $\kappa_i = k^2(m_i + m_0)$ where m_0 is the mass of the Sun. The respective differential equations for these slowly varying canonical elements read as follows

$$\begin{aligned}\frac{d\Gamma_i}{dt} &= \frac{\partial F_i}{\partial \gamma_i} \\ \frac{d\gamma_i}{dt} &= -\frac{\partial F_i}{\partial \Gamma_i}\end{aligned}\tag{2}$$

For every $\Gamma_i = (L_i, H_i, G_i)^T$ there are conjugate elements $\gamma_i = (l_i, h_i, g_i)^T$. The so-called perturbing function F_i can be developed into Fourier-series with respect to time and – because of the form of the differential equations of the elements – one gets the following expression for an element Γ_i after derivation with respect to the conjugate element and subsequent integration.

$$\Gamma_i = \Gamma_i^0 + \Gamma_i^1 t + \sum_{j,k \neq 0} \frac{E_{1,2}}{j \cdot n_1 + k \cdot n_2} \cos[(j \cdot n_1 + k \cdot n_2)t + D_{1,2}].\tag{3}$$

The amplitudes $E_{1,2}$ are polynomial expressions of the elements $a_1, a_2, e_1, e_2, i_1, i_2$ and the phase coefficients $D_{1,2}$ depend on the Delaunay elements g_1, g_2, h_1, h_2 for a planet 1 perturbed by a planet 2. The problem is the divisor $j \cdot n_1 + k \cdot n_2$ which will be small whenever the summation indices j and k are close to commensurability

$$\frac{n_1}{n_2} \approx -\frac{k}{j}.\tag{4}$$

It means whenever the two mean-motions are close to the ratio of two small integers one speaks of a Mean Motion Resonance. Then the value of $E_{j,k}$ is divided by a quantity close to zero and the amplitude of the periodically perturbing term is large. What we outlined here is a 1st order theory for two planets where the elements of the perturbing planet 2 are kept constant. For precise planetary theories one needs to go to 2nd and higher orders (see e.g. [2]).

The formal description of a MMR works also for asteroids with the constraint – which is also valid for planetary orbits – that the eccentricities and inclinations of the celestial bodies need to be sufficiently small and that the two bodies (planet-planet or asteroid to planet) do not come too close to each other. Otherwise a 1st order perturbation theory, is not valid any more.

The width of a resonance can be determined via some elementary considerations in the pendulum approximation on the phase plane already (e.g. in [13], p. 226ff). The translation into the semimajor axes versus eccentricities plane can be studied in detail e.g. in [25](p.195 ff) and in [26](p. 321ff). In the respective plot Fig. 3 we show the limiting lines of some of the resonances; they all are V-shaped and normally the largeness with respect to the semimajor axes increases with larger eccentricities. At $a \sim 2.5$ AU a very sharply bordered, empty region is visible, which corresponds to the 3:1 MMR between an asteroid and Jupiter; it will be thrown out via separatrix crossing (for details see next chapter). The empty region close to $a = 2.8$ AU is due to the 5:2 MMR. In the right half of the plot for eccentricities $e > 0.2$ one can see, that the V-shaped resonances join; in nonlinear dynamics one speaks of overlapping resonances. When a body is in such a region, it can reach quite large eccentricities, so that a consecutive close approach to a planet leads to an escape. In fact there are no asteroids which stay in this resonance overlapping region for long times; it is consequently depopulated.

2.3 Secular Resonances

In the description of orbital elements Γ_i given above we neglected that the phase coefficients $D_{j,k}$ contain – in a higher approximation – also terms which vary with time. It means that the orbital elements for the perihelion and the longitude of the nodes between two celestial bodies are not constant, but have a secular trend: $\omega_1 = \omega_1^0 + \omega_1^1 \cdot t$ and $\Omega_1 = \Omega_1^0 + \Omega_1^1 \cdot t$. As a consequence for

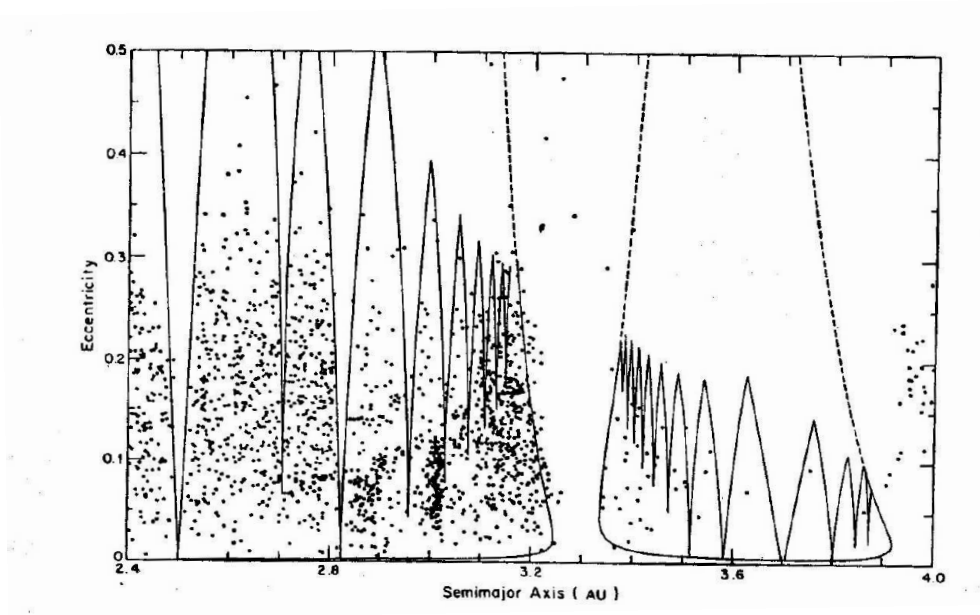


Figure 3: Resonance width of different MMR in the main belt of asteroids; eccentricity versus semimajor axes. (after [25])

a so-called secular solution – where one averages over the fast variables – the mean motion ⁴ term $j \cdot n_1 + k \cdot n_2$ disappears and instead new ones like $j \cdot \omega_1^1 + k \cdot \omega_2^1$ respectively $j \cdot \Omega_1^1 + k \cdot \Omega_2^1$ and different combinations of them appear as small divisor after the integration. The so-called Laplace-Lagrange-secular theory determines the ‘fundamental frequencies’ of the system of the orbital elements for a planet $j = 1, 8$

$$h_j = e_j \sin \tilde{\omega}_j \quad (5)$$

$$k_j = e_j \cos \tilde{\omega}_j \quad (6)$$

$$p_j = i_j \sin \Omega_j \quad (7)$$

$$q_j = i_j \cos \Omega_j \quad (8)$$

where $\tilde{\omega}_j = \Omega + \omega$. The solution is given in the following form:

$$h_j = \sum_{i=1}^2 e_{ji} \sin(g_i t + \beta_i) \quad (9)$$

$$k_j = \sum_{i=1}^2 e_{ji} \cos(g_i t + \beta_i) \quad (10)$$

$$p_j = \sum_{i=1}^2 I_{ji} \sin(f_i t + \gamma_i) \quad (11)$$

$$q_j = \sum_{i=1}^2 I_{ji} \cos(f_i t + \gamma_i). \quad (12)$$

⁴which is equivalent to keeping the semimajor axis constant

Table 2: Basic data from a secular theory of the planets (after [25])

planet	g_i ("/yr)	f_i ("/yr)
Mercury	5.5964	-5.6174
Venus	7.4559	-7.0795
Earth	17.3646	-18.8512
Mars	17.9156	-17.7482
Jupiter	4.2575	-
Saturn	28.2455	-26.3450
Uranus	3.0868	-2.9927
Neptune	0.6726	-0.6925

The frequencies g_i and f_i are the eigenvalues of special matrices not defined here in detail (for details see [25] and [26]) and e_{ji} and I_{ji} are the components of the corresponding eigenvectors. The frequencies given in Table 2 are good approximations to the above mentioned ω_j^1 and Ω_j^1 for the 8 planets; from these values one can derive the corresponding periods.

For minor planets (NEAS, main belt asteroids and Kuiperbelt objects) these values of the planets combined with the motions of the perihelia (nodes) of the asteroid may cause also small divisors and consequently large perturbations on the elements. This is visible in Figure 4, where different secular resonances are acting. ν_i stands for a SR in the perihelion of an asteroid with the i -th planet, ν_{1i} for the SR in the motion of the node. One can see the empty regions corresponding to these resonances. The 'proper elements' are a kind of mean values of the elements which are more or less constant (for details see [25] or [26]).

2.4 The Kozai resonance

For asteroids and comets, which can be regarded massless compared to the planets, another interesting resonance appears to be of importance for their dynamics. It was Kozai (1962) who showed, that an asteroid perturbed by a Jupiter in an almost circular orbit ($e_{jup} \leq 0.06$) does not suffer from secular changes in its semimajor axis, because in this case in the Delaunayelement H defined above, the semimajor axis a remains constant, whereas the inclination i and the eccentricity e of the asteroid are coupled. As a consequence for a value of H given for $e = 0$ there is a maximum inclination i_{max} and vice versa a certain value e_{max} for every inclination:

$$\begin{aligned} i_{max} &= \arccos \frac{H}{\sqrt{a}} \\ e_{max} &= \arccos \sqrt{1 - \frac{H^2}{a}} \end{aligned} \quad (13)$$

In a detailed analysis one can show (see [25], p. 157ff) that above a certain threshold i_{max} the point $e = 0$ becomes an unstable equilibrium point and the phase space is divided into three separate parts. In two regions (Fig. 5) the orbital element ω (the Delaunay-element g) librates around the two stable equilibrium points $g = 90^\circ$ and $g = 270^\circ$. The picture shown as an example is qualitatively the same for all ratios of the semimajor axis $a_{ast}/a_{jup} < 1$. It looks different for Kuiperbelt objects due to the perturbations of Neptune, but it is also a cause of unstable motion for celestial bodies located in the outer part of the Solar System.

This coupling of the i and e of an asteroid for large inclinations is responsible for unstable motion not only for asteroids but also for comets to become sun-grazers, which was concluded by Bailey et al (1992). The eccentricity which sometimes is quite large, allows close approaches to a

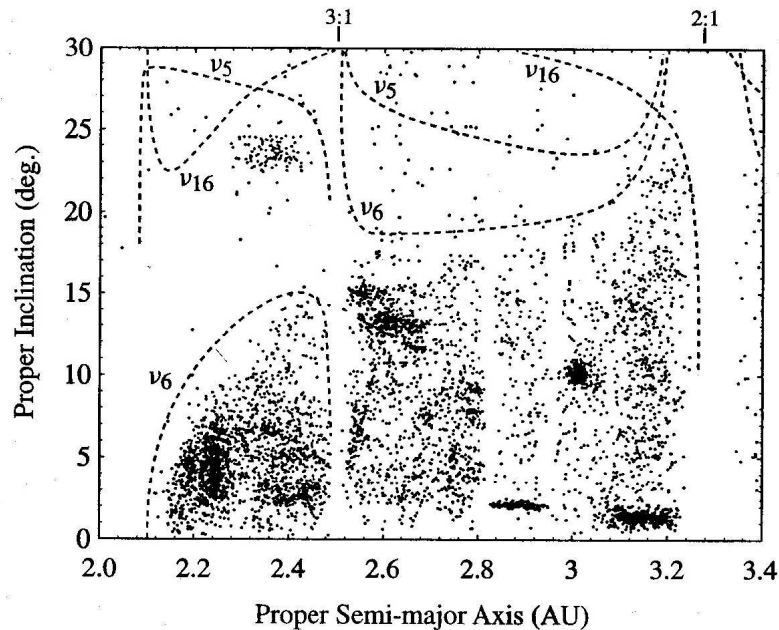


Figure 4: Secular Resonances in the main belt of asteroids; inclination versus semimajor axes. The empty regions for $a \sim 2.5$ AU and $a \sim 2.8$ correspond to the 3:1 respectively to the 5:2 MMR (from [25]).

planet which then ejects the celestial body from its orbit. The theory developed by Kozai is not only valid for one perturber, but it was extended to all four large planets by [21].

2.5 The Three Body Resonances

In the discussion of MMR we explained the appearance of a small divisor (see equation (3)) which will lead to large perturbations. Recently it was found out, that also three-body MMR may play an important role for the dynamics of asteroids. The development of the perturbing function can be extended already to 1st order taking into account also Saturns perturbations besides the ones of Jupiter for the motion of an asteroid. Thus a term $j \cdot n_1 + k \cdot n_2 + l \cdot n_3$ appears, where j, k and l are summation indices and the mean motion n_1 corresponds to the asteroid, n_2 to Jupiter and n_3 to Saturn. For certain values of the mean motion of the asteroid combined with the mean motions of Jupiter and of Saturn these values may be close to 0 and cause also large variations in the perturbed orbital element of the asteroid (e.g. [16]).

$$j \cdot n_1 + k \cdot n_2 + l \cdot n_3 \sim 0. \quad (14)$$

In Table 3 we show how TBR are present for asteroids in the mainbelt. They are much denser than the normal MMR but they are less strong; nevertheless they are essential when one tries to understand the dynamics of some asteroid like Veritas [31].

3 Planets and Satellites

In our planetary system MMR resonances are frequently to be, which can be seen in Table 4 for the planets and for the satellite systems of the giant planets. We shortly describe the well known

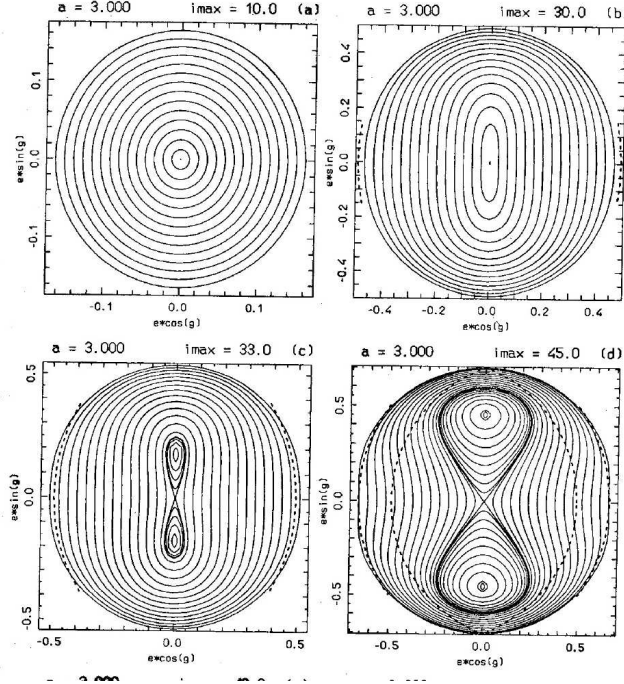


Figure 5: The Kozai Resonances for a semimajor axes $a = 3$ AU of an asteroid in polar coordinates where $g = \omega$. Note the from $i = 30^\circ$ on two stable equilibrium points arise around $g = 90^\circ$ and $g = 270^\circ$.

Table 3: 3 body MMR

name	number	a [AU]	n_{jup}	n_{sat}	n_{ast}	divisor
Educatio	2440	2.2157	4	-1	-1	0.2990
Lola	463	2.3977	4	-2	-1	0.2655
Tristan	1966	2.4476	7	-2	-2	0.5149
Somalia	1430	2.5599	7	-3	-2	0.4815
Tyche	258	2.6155	2	2	-1	0.2328
Kalypso	53	2.6190	6	-1	-2	0.4601
Metcalfia	792	2.6230	4	-3	-1	0.2322
Lena	789	2.6857	7	-4	-2	0.4481
Genua	485	2.7525	3	-1	-1	0.2158
Kalliope	22	2.9095	4	-4	-1	0.1891
Emanuela	576	2.9860	5	-1	-2	0.3820
Aho	2395	3.0790	3	-2	-1	0.1825
Kunigunde	936	3.1385	6	2	-3	0.5654
Hygiea	10	3.1418	8	-4	-3	0.4364
Dione	106	3.1708	3	3	-2	0.3477
Veritas	490	3.1738	5	-2	-2	0.3486
Turandot	530	3.2080	7	-2	-3	0.5147

Table 4: Mean motion resonances in the Solar System (*the * denotes exact mean motion resonances*)

System	Resonances			
Solar System	Jupiter–Saturn	5:2	Saturn–Uranus	3:1
	Uranus–Neptune	2:1	Neptune–Pluto*	3:2
Jupiter System	Io–Europa*	2:1	Europa–Ganymed*	2:1
Saturn System	Mimas–Tethys*	2:1	Enceladus–Dione*	2:1
	Dione–Rhea	5:3	Titan–Hyperion*	4:3
Uranus System	Miranda–Umbriel	3:1	Ariel–Umbriel	5:3
	Umbriel–Titania	2:1	Titania–Oberon	3:2

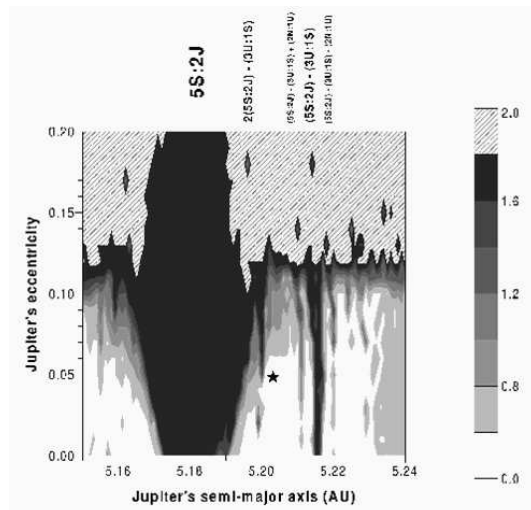


Figure 6: The 5:2 MMR for Jupiter and Saturn: eccentricity versus different semimajor axis of Jupiter indicating the state of chaoticity: black means chaotic, white denotes stable orbits (after [23]).

so-called **great inequality** between the giant planets Jupiter and Saturn, where the mean motions are $n_{\text{Jup}} = 0.^{\circ}08309/\text{day}$ and $n_{\text{Sat}} = 0.^{\circ}03346/\text{day}$ which is close to the ratio 5 : 2. The perturbations acting between both planets have a small divisor

$$2n_{\text{Jup}} - 5n_{\text{Sat}} = 0.^{\circ}00112 \quad (15)$$

which leads to a period of $360^{\circ}/0.^{\circ}00112$ days corresponding to about 880 years. In fact this perturbation is rather large and causes a variation in the amplitude in Jupiter's longitude of almost $20'$ and for Saturn $48'$ with the above mentioned period. Numerical experiments have shown that the closeness to the 5:2 MMR make the system Jupiter Saturn 'marginally stable'. Only a relative small change in the present orbital elements would shift the motion into a zone of chaos (e.g. [32], [22] and [30]).

Given the actual value of the semimajor axis of Jupiter ($a = 5.202$) this shift of only 1 percent to smaller values would put it exactly into the 5:2 MMR where relatively large perturbations (Fig. 6 and Fig. 7) are acting. In the semimajor axis – eccentricity plot this is visualized as black V-shaped chaotic region, where the spectral number as chaos indicator has been used (see [23] for details). In the semimajor axis – inclination plot one can see it from the large values of the eccentricities). The V-shaped structure of MMR in both plots is also present in the respective graph for the asteroid

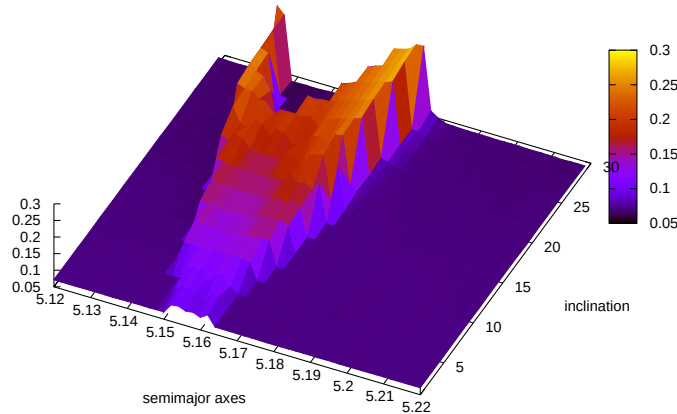


Figure 7: The 5:2 MMR for Jupiter and Saturn where we plotted the maximum eccentricity of an orbit (z-value) during the integration of 10 Myrs with respect to the initial inclination and semimajor axis of Jupiter.

(see Fig. 3). the resonance with larger inclinations which is also present for the semimajor axis – eccentricity plot (see Fig. 3).

Quite remarkable is the 3:2 MMR between Neptune and Pluto: although Plutos closest distance to the Sun is $r = 29.7 \text{ AU}$ ⁵ and thus Pluto approaches the Sun more closely than Neptune, the two planets never come close. Whenever Pluto is in its perihelion Neptune is opposite to the Sun!

Another example, usually not mentioned in any treatise of solar system dynamics, is the 13:8 MMR of Venus and Earth (Fig. 8). A somewhat larger eccentricity of the Earth could lead to larger eccentricity perturbations of Venus and Earth.

4 Asteroids and Comets

4.1 Main belt asteroids

The role of MMR can be seen in the structure of the main belt asteroids moving between Mars and Jupiter (see Fig. 9). For some of the resonances the number of asteroids is quite small; one can distinguish the 4:1, the 3:1, the 5:2 and the 2:1 MMR in this plot which form the so-called Kirkwood gaps, named after the astronomer Daniel Kirkwood (1814 - 1895) who first discovered them. He did not have the correct explanation for their existence, he just concluded that they are connected with large perturbations. The correct answer was given only 100 years later when J. Wisdom [33] discovered that the depletion in the 3:1 MMR is due to the acting of chaos with Jupiter on an elliptic orbit. The asteroid seems to stay with a small eccentricity for millions of years but then 'suddenly' the eccentricity increases significantly. This somewhat strange behaviour can be explained by a crossing of the separatrix between two qualitatively different motions, the libration leading to small eccentricity changes and the circulation leading to large eccentricities. These large values of the eccentricity allow close approaches to the planet Mars, which lead to a subsequent ejection of the asteroid from the 3:1 MMR. In principle this procedure works for all the gaps visible in Figure 9.

⁵which is caused by its large eccentricity of $e=0.25$ in spite of a semimajor axis of $a = 39 \text{ AU}$

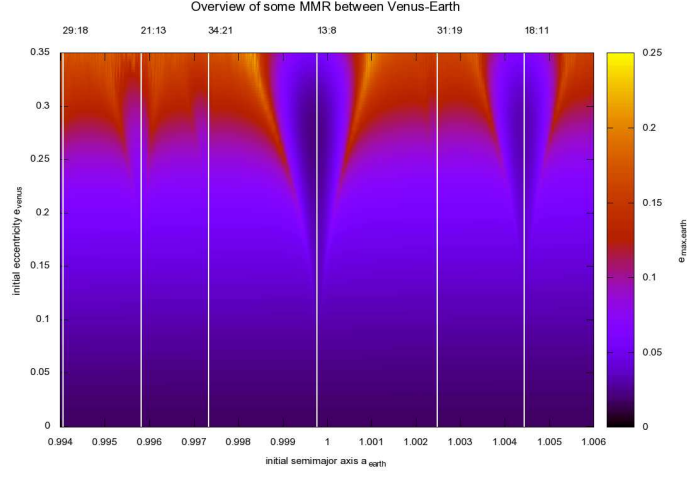


Figure 8: The 13:8 MMR of Venus and Earth and the close by high order MMR: the semimajor axis of the Earth is plotted versus the eccentricity of Venus. The color indicates the maximum eccentricity achieved during the integration time of 1 Myr.

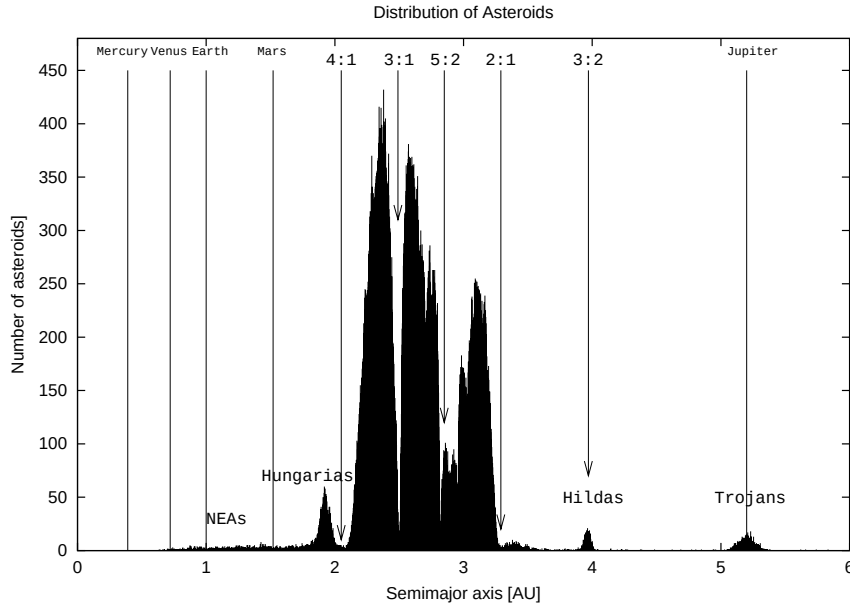


Figure 9: The distribution of the asteroids in the inner Solar System in the so-called main belt between Mars and Jupiter. Note that some resonances are depleted, others, like the Trojans in 1:1 MMR are populated with a large number of bodies

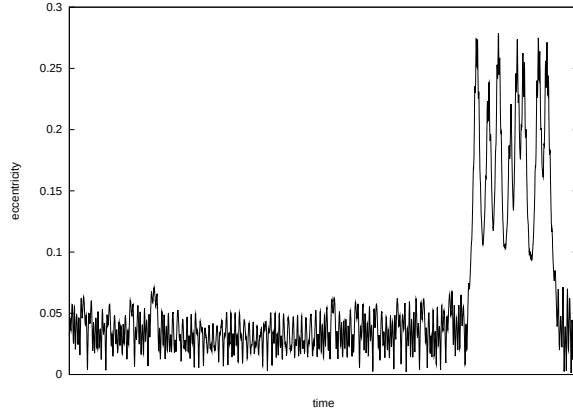


Figure 10: The dynamical evolution of the eccentricity of an asteroid in the 3:1 MMR with Jupiter over 10^6 years in the elliptic restricted three body problem. For quite a long time there are only small variations of the ellipticity of the orbit up to a sudden increase after some $8 \cdot 10^5$ years.

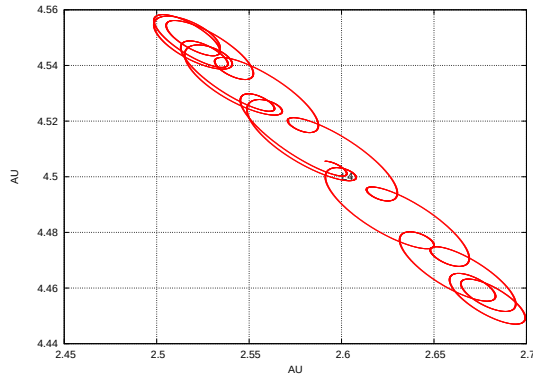


Figure 11: Libration motion in the vicinity of the Lagrange equilibrium point L_4 of an asteroid in the 1:1 MMR with Jupiter in a rotating coordinate system.

4.2 The Trojans

The group of the Jupiter Trojans are in a 1:1 MMR and today we have knowledge of more than 2000 Sun orbiting small bodies close to the Lagrangian points L_4 and L_5 (compare Fig. 9). These equilibrium points are 60° ahead and 60° behind Jupiter in its orbit. The stable librating motion has two well distinct orbital periods, where one corresponds almost to the one of Jupiter of 12 years, the other one is about 159 years. In the corresponding plot (Fig. 11) one can see the two different motions in the framework of a rotating coordinate system where Jupiter has a fixed position (for details see e.g. [5] or [6]). We need to mention, that Neptun also has Trojans asteroids (up to now only 5 have been observed); Saturn's and Uranus' Trojans seem to be unstable [5].

In the Figure 12 we show the results of a long term integration (10^7 years) of Trojans close to the Lagrangian point L_4 in the dynamical model of the outer Solar System (Jupiter, Saturn, Uranus and Neptune) where the mass of the inner planets was added to the mass of the Sun. The location of L_4 in a rotating coordinate system (the same as in Figure 11) is $a = 5.2$ in the x-coordinate and 60° in the y-coordinate. The maximum eccentricity is plotted in the z-axis; the stable region is well defined, as one can see from the sharp edges confining this zone.

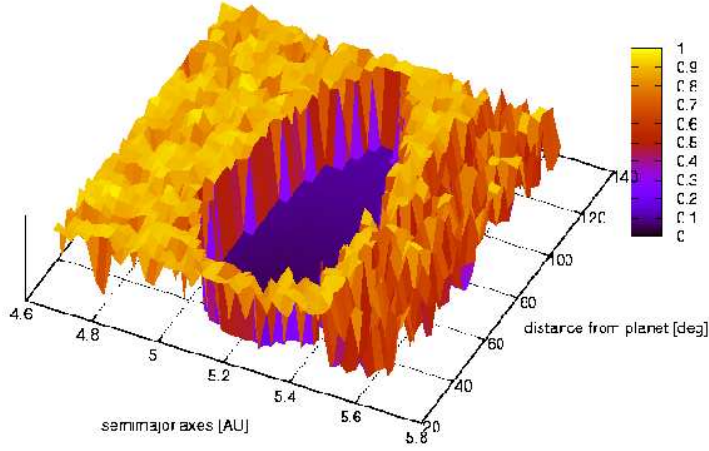


Figure 12: Extension of the stability region around the Lagrangian point L_4 in semimajor axes and in angular distance from L_4

4.3 Near Earth Asteroids

The Near Earth Asteroids (=NEAs) are a group of asteroids which – from time to time– come close to the Earth – or even have collisions with our planet. Such a close approach is visible from the upper plot Fig. 13 where an asteroid suffers from a close encounter with the Earth after 20000 years which causes a drop of the semimajor axis from $a \sim 1$ AU down to $a \sim 0.6$ AU. The different horizontal lines in the development of a show the temporary 'capture' into high order resonances. Quite remarkable is the capture into the 1:1 MMR with the Earth by Nereus (lower graph) between $3.5 \cdot 10^5 < t < 5 \cdot 10^5$. Another interesting feature is that Asclepius is captured into the Kozai resonance (see section 3.2) several times in its dynamical evolution. Very well visible is the inverse quantitative behaviour for the elements inclination and eccentricity between $2.2 \cdot 10^5 < t < 3.3 \cdot 10^5$. It should just be mentioned that also short periodic comets are quite often in MMR with Jupiter where they are captured for certain time spans.

4.4 The Edgeworth-Kuiper belt

Part of the large population of asteroids with semimajor axes $a > 39$ AU is the group of the Plutinos, which are asteroids in a 3:2 resonance with Neptune like Pluto itself (up to now we have evidence for several hundreds of these bodies⁶). The structure of this 3:2 MMR is shown in Fig. 14 where the different values of the libration angle σ defined as

$$\sigma = 2 \cdot \lambda_{Neptune} - 3 \cdot \lambda_{plutino} + \tilde{\omega}_{plutino} \quad (16)$$

are plotted as lines in the semimajor axis – eccentricity diagramm. It is obvious that in the center of resonance this libration is quite small (only 20°), then it increases with the distance but

⁶these numbers significantly increase every year

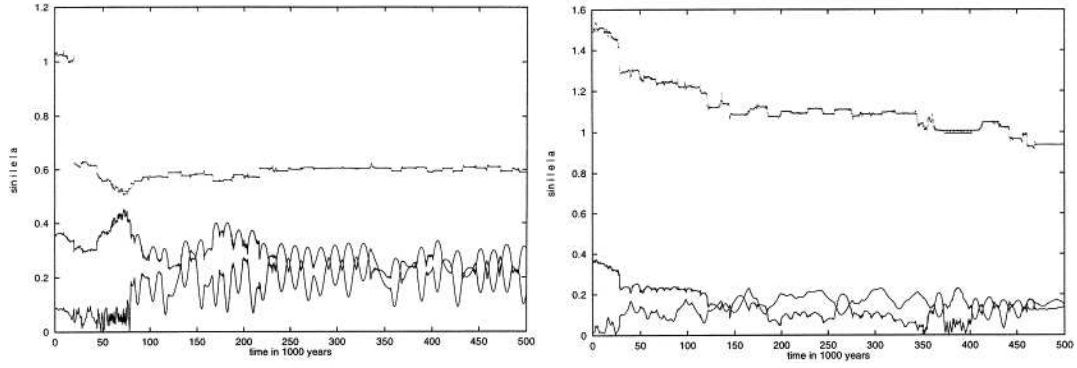


Figure 13: The evolution of the orbital elements semimajor axes (dots), eccentricities and $\sin(i)$ (lines) of Asclepius (left graph) and Nereus (right graph) for $5 \cdot 10^5$ years (after [3])

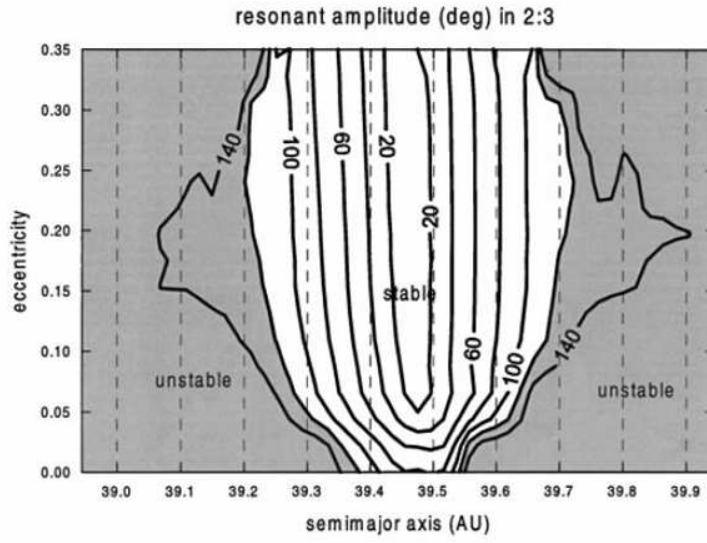


Figure 14: The 3:2 MMR eccentricity versus semimajor axis. The libration angle (solid lines) of the plutinos inside the stable (white) region are shown.

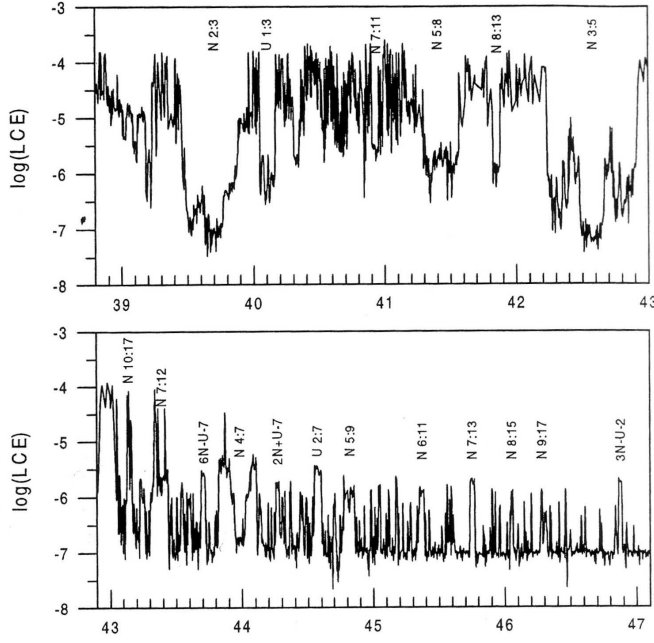


Figure 15: Resonances in the Edgeworth-Kuiperbelt between $39AU \leq a \leq 47AU$. The Lyapunov Exponent (LCE) characterizes stable (LCE small) and unstable motion (LCE larger). Several TBR are visible in the outer region marked e.g. at $a = 43.7AU$ as 6N-U-7; for detailed information see the text.

stays almost independent of the eccentricity. The exception are small values of the eccentricity ($e \leq 0.05$) where also large variations (up to 80°) are possible. The limit of stability for a plutino is a libration angle of 120° ; then, obviously the protection mechanism (for a perihelion of the plutino Neptune is opposite to the Sun) does not work any more and the asteroid is on an unstable orbit.

Even the outer region ($a \geq 40$ AU) which is called the Edgeworth-Kuiper belt (=EKB)⁷ where many bodies are moving which have sizes up to the one of Pluto and larger⁸ the motion suffers from many MMR and also TBR. show In Fig. 15 we show for fictitious bodies the computed Lyapunov exponents (LCE), which are a well known measure of the chaoticity of an orbit. A small value indicates regular stable motion, a large value shows that the respective body is not on a regular orbit. Exactly in the 2:3 MMR, where Pluto and the Plutinos are moving, the very small LCEs are well visible. Also around $a = 42.5$ AU, in the 3:5 MMR the LCE is quite small and indicates stable regular motion. In fact many objects are moving in this resonance. At any rate it is clear that both, MMR and TBR are important for an understanding of its dynamical structure. So we see e.g. around $a = 46.9$ AU (Fig. 15) a TBR marked as 3N-U-2 which means the small divisor created by $3n_{Neptune} - n_{Uranus} - 2n_{object}$. A detailed investigation shows that SR and KR are also present in the EKB.

⁷after K. E. Edgeworth (1880-1972) and G. P. Kuiper (1905-1973), who predicted a disk of celestial bodies and dust some 60 years ago

⁸Quaoar has a diameter of 1300 km, Sedna is estimated to have a diameter of 1800 km

5 Retrograde Orbits

In the satellite systems of the planets Jupiter or Saturn there exist many retrograde small moons which turn out to be stable than prograde ones. From the computation of periodic orbits and from numerical integrations of fictitious bodies this fact is confirmed. Theoretical work concerning this question has been undertaken already some twenty years ago (eg. [12]). New considerations of possible retrograde orbits (in extrasolar planetary systems [11]) showed that out of 5000 randomly distributed orbits for three systems in the 2:1 MMR and two in the 5:1 MMR only about 20 percent were stable in the prograde case, but half of them in the case when they are in a retrograde MMR. These interesting results pushed us to a study of retrograde orbits of the main belt asteroids between Mars and Jupiter as a toy model. The results are shown in Fig. 16 where this region is investigated with respect to stable orbits for inclinations $0^\circ \leq i \leq 180^\circ$. In the region up to approximately $i = 25^\circ$ the 3:1, the 5:2 and the 2:1 MMR are visible as small hills. On the other hand for the retrograde orbits in the range $160^\circ \leq i \leq 180^\circ$ the 3:1 and the 5:2 MMR are not visible by means of larger perturbations (larger eccentricities). The exception is the 2:1 MMR, which is even stronger than for prograde orbits. More investigations from the point of the theory of perturbations and numerical experiments need to be undertaken and are in progress.

6 Conclusions

In this short review I wanted to show how the different resonant motions of Solar System bodies are influencing and even dominating the dynamical structure. The most obvious is the MMR, where the gas giants Jupiter and Saturn, which are in a 5:2 resonance, as quite good an example and the 3:1 Kirkwood gap in the main belt of asteroids. Especially for small bodies it is important to include them in the considerations, which we have shown for the NEAs, as they are often temporarily captured into high order resonances with the planets. But they also suffer from the SR which are caused by the nonconstancy of the nodes and perihelions of the planets. The main belt of asteroids is also sculpted by the different MMR, but also from SR especially for highly inclined orbits. The belt of celestial bodies outside Neptune's orbit has a structure which is essentially dominated by resonances of all different kinds. In general retrograde resonances – as has been shown by numerical experiments – don't seem to be able to destabilize an orbit. As a concluding remark we can say that the architecture of our planetary system is structured the way it is by the presence of resonances.

7 Epilog: Extrasolar Planets

Some 15 years ago the first extrasolar planetary system was discovered around a pulsar [34]. Soon there after, the first planet around a sun-like star [20] was detected via radial velocity measurements and nowadays more than 300 planets are known around other stars ⁹. The big surprise during these first years of discovery was, that many of the planets seem to be on very eccentric orbits (25%, 70 out of almost 280 planets discovered by radial velocity methods have an eccentricity $e > 0.35$) ¹⁰. In the respective Figure 17, eccentricities versus the observed semimajor axes for single planetary systems (upper graph) and multiplanetary systems (lower graph), it is obvious that single planets have larger orbital eccentricities. In a recent publication [1] it is claimed that for many of the observed single star systems a fit with two planets in 2:1 MMR would give the same observed radial velocity curve. The interesting point is that then the masses of these two

⁹<http://exoplanet.eu/> the extrasolar planets homepage maintained by J. Schneider, Observatory of Paris since 1995

¹⁰which is not true for photometrically discovered planets passing across the disk of the host star. These transits can only be observed with a certain probability for planets with small semimajor axes and consequently small eccentricities

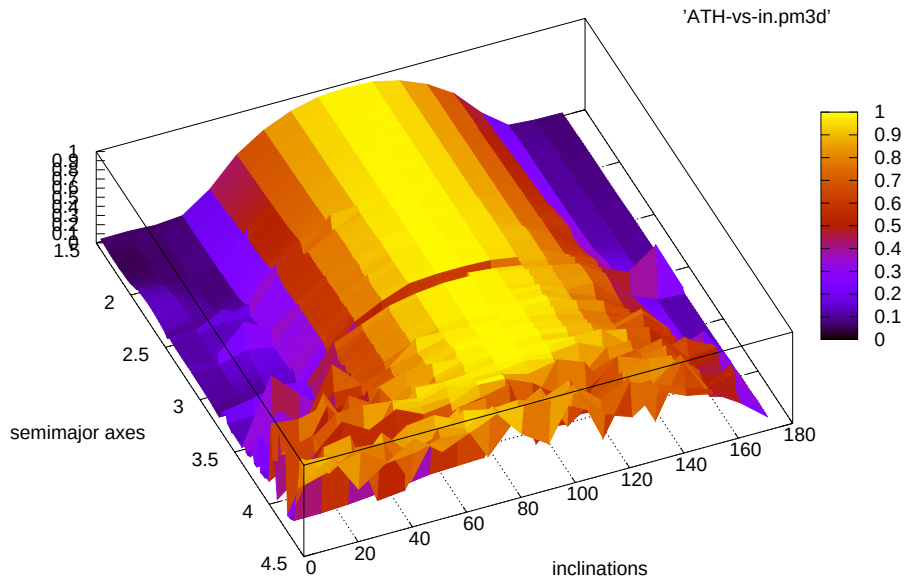


Figure 16: Orbits in the mainbelt of Asteroids between Mars and Jupiter with respect to their inclination for $0^\circ \leq i \leq 180^\circ$

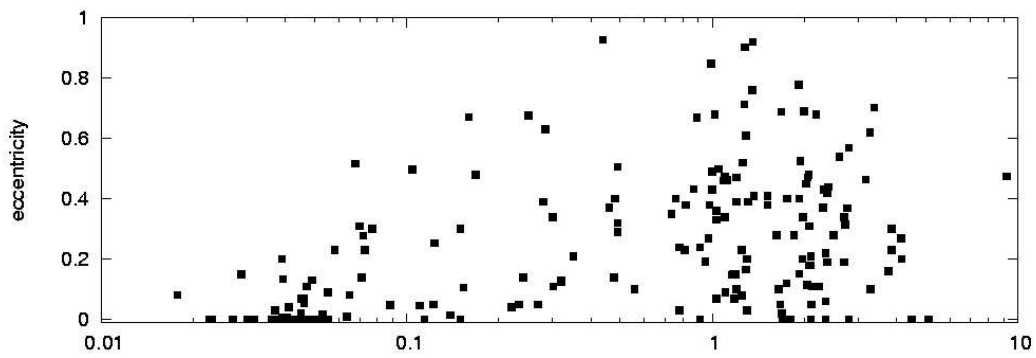


Figure 17: Eccentricities of exoplanets versus semimajor axes (data from J.Schneider: <http://exoplanet.eu>)

Table 5: Extrasolar Planets in mean motion resonances.

PLANET	m_{Sun}	m_{jup}	P (days)	a (AU)	e (+/-)	MMR	status
GL 876 b	0.32	1.06	29.995	0.1294	0.314 (0.02)	2:1	al
GL 876 c	0.32	3.39	62.092	0.2108	0.051 (0.02)	2:1	al
HD 82943 b	1.05	0.88	221.6	0.73	0.54 (0.05)	2:1	al
HD 82943 c	1.05	1.63	444.6	1.16	0.41 (0.08)	2:1	al
55 Cnc b	1.03	0.83	14.65	0.115	0.03	3:1	no
55 Cnc c	1.03	0.20	44.27	0.241	0.41	3:1	no
47 UMa b	1.03	2.54	1089	0.061 (0.014)	5:2	7:3?	
47 UMa c	1.03	0.76	2594	0.1	(0.1)	5:2	7:3?

planets would be significantly smaller than the ones published now; some of them could be even planets comparable in mass with the Earth. Many of the planets are even larger than Jupiter, and, that was and is the second difficult problem for astronomers to understand, many of these gas giants (called hot Jupiters) have very close by orbits to the host star (down to a tenth of Mercury's semimajor axis).

The motions in resonance occur for some planets like two planets in the 5 planets system 55 Cancri (the innermost planets are in the 3:1 MMR) and, as another example in Gl 876 where the planets are in a 2:1 MMR. An interesting fact is that for the latter example (see Table 5) the large values of the eccentricities would lead to close encounters if not a special mechanism – called 'apsidal locking' – would prevent this. Under this terminus astronomers understand that the perihelia (orbital elements ω) of these two orbits are either aligned or antialigned (for details see e.g. [24]).

It should be stressed finally, that from new discoveries like the ones by CoRoT and future space observations we expect new exciting results. The satellite mission CoRoT ¹¹. is in part dedicated to observe planets via transits. But also from ground based discoveries (the Extremely Large Telescope (ELT) will have a 42m mirror) we should be able to answer old questions concerning the formation of planetary systems in general and their architecture, but – as always – new questions will come up.

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¹¹Convection, Rotation & planetary Transits) was initiated by the French space agency, CNES and it is supported from the European Space Agency (ESA), Austria, Brazil, Belgium, Germany and Spain. The satellite was launched in late December 2006 into a polar circular orbit of 896km altitude and is planned to work for 2 1/2 years at least

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