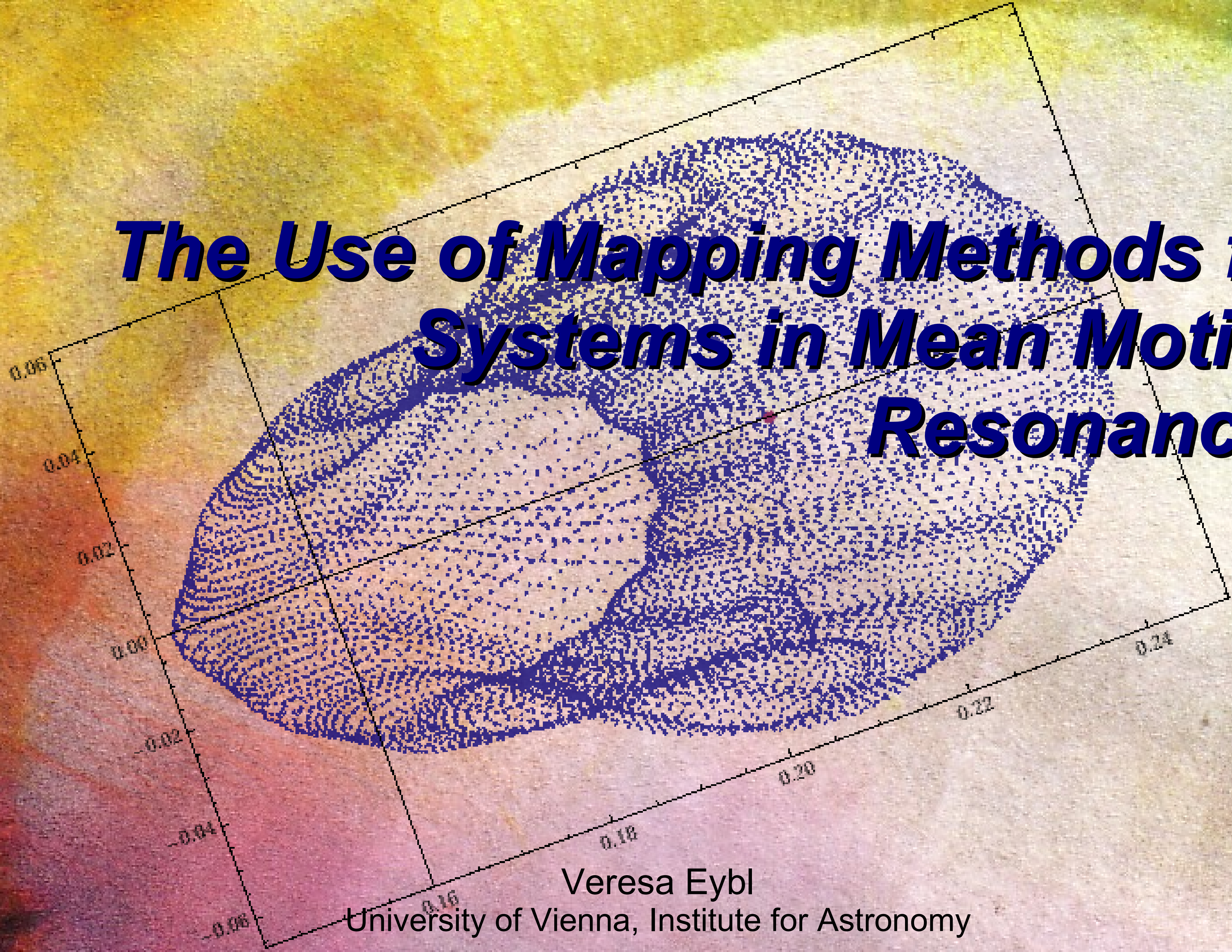

The Use of Mapping Methods for Systems in Mean Motion Resonances

Veresa Eybl
University of Vienna, Institute for Astronomy

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Introduction

- A mapping presents a way to study the behavior of nonlinear dynamical systems
- Semi-analytical method
- Advantages over numerical integration:
  - Computing time
  - Accuracy
- Phase space topology of the mapping and the actual dynamical system has to be the same

Basic recipe

Take a nearly integrable Hamiltonian system:

0. The R3BP
1. construct the disturbing function
2. average the DF
3. expand the Hamiltonian around the resonance
4. introduce semi-cartesian coordinates & center shifting
5. construct the mapping equations
6. find a suitable Surface of Section

0. The elliptic restricted 3BP

- We consider a system of 2 massive bodies (primary and secondary) and a third massless body
- Good approximation for asteroid motion
- 3BP is not solvable analytically, but it can be represented by a nearly integrable Hamiltonian system
- Equations of motion of the test particle consist of an unperturbed, Keplerian part due to the gravitational attraction of the central body, and a perturbed part due to the attraction of the second body (planet etc.)

1. The Disturbing Function

- $R = \frac{1}{\Delta} - \frac{1}{r} + \frac{1}{2}\frac{\Delta^2}{r'^3} - \frac{1}{2}\frac{r^2}{r'^3}$

- Express r, r', Δ in terms of Keplerian elements
- Use the following relations

$$r = a(1 - e\cos E)$$
$$M = E - e\sin E$$

- Bessel functions: $\frac{\mathrm{d}E}{\mathrm{d}M} = \frac{1}{1 - e\cos E} = \frac{a}{r}$

$$\frac{a}{r} = 1 + 2\sum_{s=1}^{\infty} J_s(se)\cos(sM)$$

1. The Disturbing Function

- Mean motion resonance $\frac{p}{p+q},\ q \ldots order\ of\ resonance$

- Taylor series expansion of all expressions up to the q-th order in e

$$r, r', r^{-1}, r'^{-1}, rr' \rightarrow f\left(e\right) + O^{q+1}$$

- $R\left(a, e, \cancel{i}, \lambda, \omega, \cancel{\Omega}, a', e', \cancel{i'}, \lambda', \omega', \cancel{\Omega'}, \right)$

$$a' \rightarrow 1,\ e' \rightarrow e_{Planet},\ \omega' \rightarrow 0$$

- $R\left(a, e, \lambda, \omega, \lambda'\right)$

2. Averaging

- Introduce modified Delaunay variables

  (after Tsiganis, 2007)

$$\lambda = \lambda, \Lambda = \sqrt{\mu' a}$$
$$\gamma = -\varpi, \Gamma = \sqrt{\mu' a}\left(1 - \sqrt{1 - e^2}\right))$$

- Averaging the disturbing function over the motion of the disturbing body (= planet) λ'

$$R(\Lambda, \Gamma, Z, \lambda, \gamma, \zeta, \lambda') \rightarrow \bar{R}(\Lambda, \Gamma, Z, \lambda, \gamma, \zeta)$$

- Construct Hamiltonian

$$H = H_0 - \mu \bar{R}$$
$$H_0 = -\frac{\mu'^2}{2p\Psi^2} - n'(p+q)\Psi$$

3. Expansion around the resonance

- location of the resonance $$a_{res} = a' \left(\frac{p}{p+q}\right)^{2/3} (1-\mu)^{1/3}$$
- resonant canonical variables
$$\psi = p\lambda - (p+q)\lambda', \Lambda = p\Psi$$
$$\psi' = \lambda'$$
$$\phi = \gamma, \Gamma = \Phi$$
$$\theta = \zeta, Z = \Theta$$
- resonant angle
$$\Psi_{res} = \frac{1}{p}\sqrt{\mu' a_{res}}$$
- New momentum $J = \Psi - \Psi_{res}$
- Hamiltonian $H = H(\phi, \Phi, \psi, J)$

4. Semi-cartesian coordinates

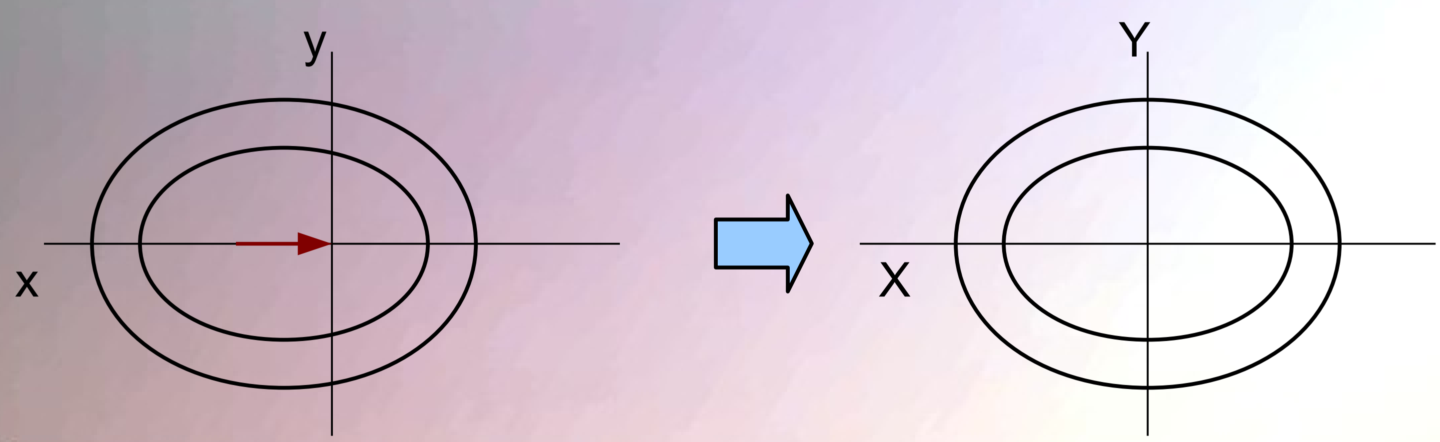
- Introduce coordinates x,y

$$\cos\phi = \frac{x}{\sqrt{2\,\Phi}},\ \sin\phi = \frac{y}{\sqrt{2\,\Phi}}$$

$$\Phi = \frac{x^2+y^2}{2}$$

- Center shift
  - Check if Hamiltonian is centered
  - Translate coordinate system if necessary $x \to X,\ y \to Y$

  => centered in origin of coordinate system



4. Mapping Equations

$$H = H(X, Y, \psi, J)$$

Hadjidemetriou's method (1993)

- Create generating function $W = W_0 + 2\pi \bar{H}$, $W_0 = J\psi + XY$
- Mapping equations

$$\frac{\partial W}{\partial J_{n+1}} = \psi_{n+1}, \ \frac{\partial W}{\partial \psi_n} = J_n$$
$$\frac{\partial W}{\partial Y_{n+1}} = X_{n+1}, \ \frac{\partial W}{\partial X_n} = Y_n$$

- $(X_n, Y_n, \psi_n, J_n) \rightarrow (X_{n+1}, Y_{n+1}, \psi_{n+1}, J_{n+1})$

5. Surface of Section

- We consider only the elliptical 2dimensional case
- The resulting mapping is 4dimensional
- To represent the 4D phase space on a 2D surface, a Poincaré surface of section has to be found
- Difficulty: finding the ideal section
  - First criterion (Tsiganis , 2007)

    $$\psi - \tilde{Q} = \pi$$

  - Second criterion

    $$J > 0$$

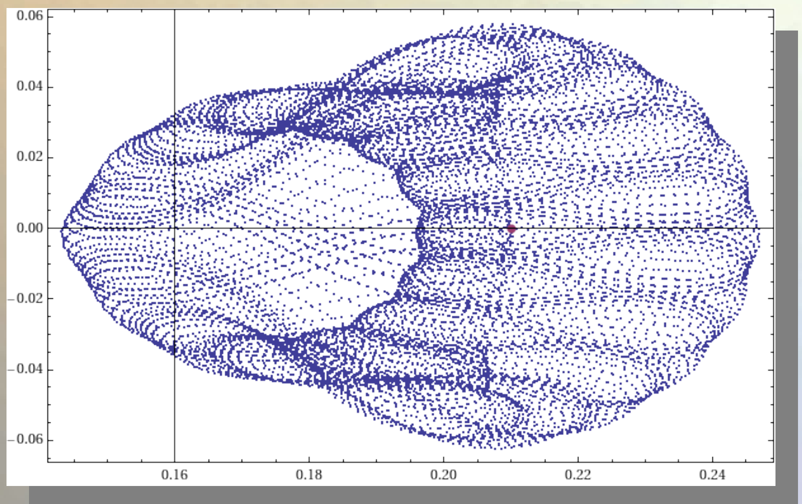
5. Surface of Section

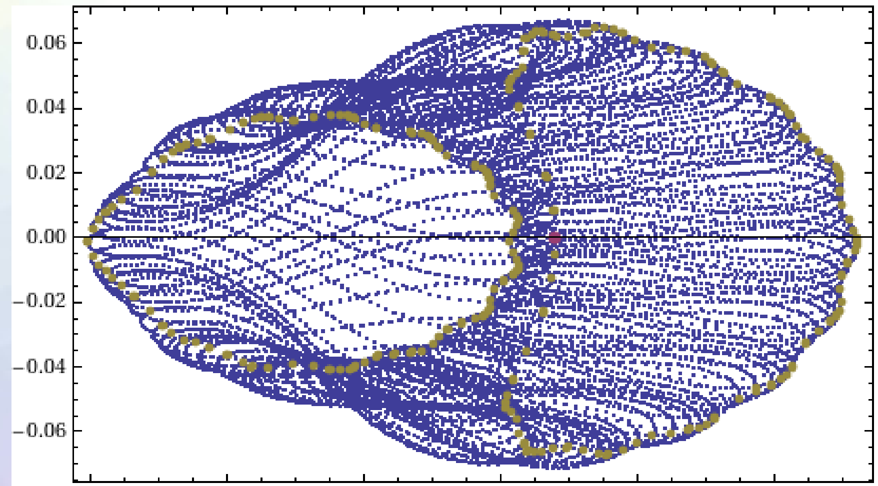
- Define the S.o.S. criterion
- Neglect 3rd dimension
- $H = H(\phi, \Phi, \psi, J)$
- Write Hamiltonian in the form

$$H = \frac{1}{2}\beta J^2 - \mu d\Phi - \mu \tilde{D}(\phi, \Phi)\cos(\psi - \tilde{Q}(\phi, \Phi))$$

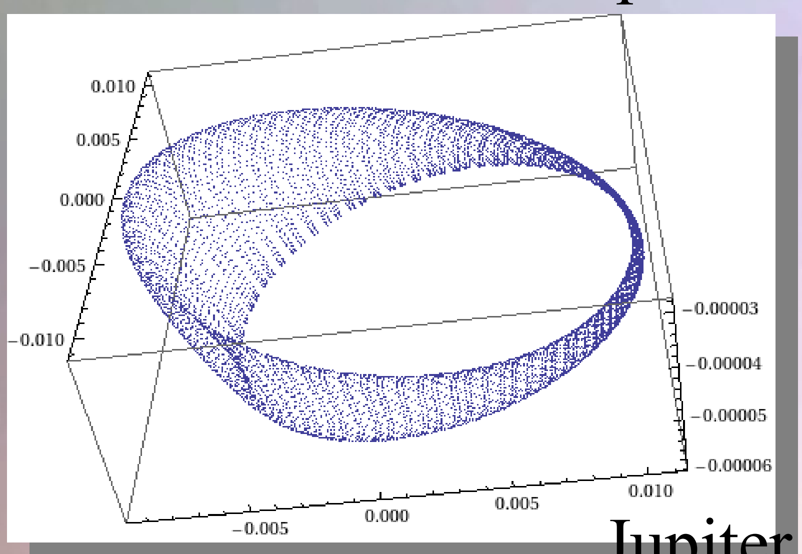
- $\tilde{D} = \sqrt{A^2 + B^2}, \tilde{Q} = 2\,arctan(\frac{-A+\sqrt{A^2+B^2}}{B})$
  $A = \sum D\,cos(k\phi), B = \sum D\,sin(k\phi)$

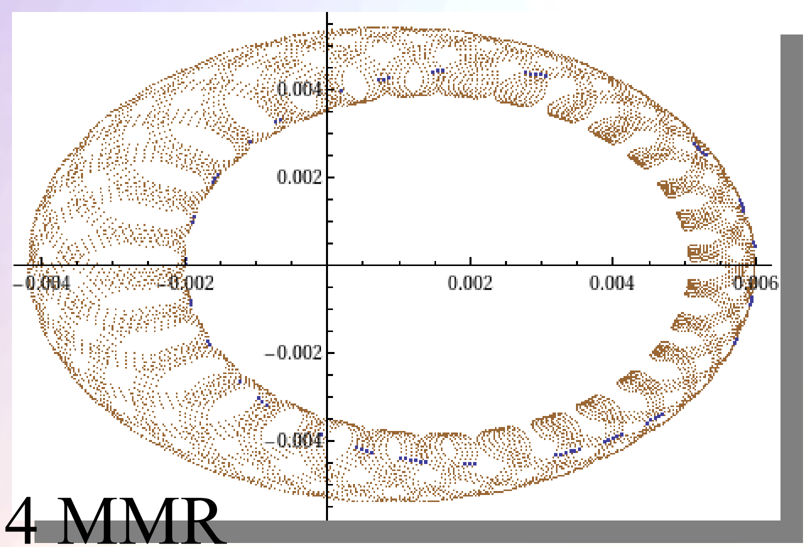
Projections into 3D and 2D





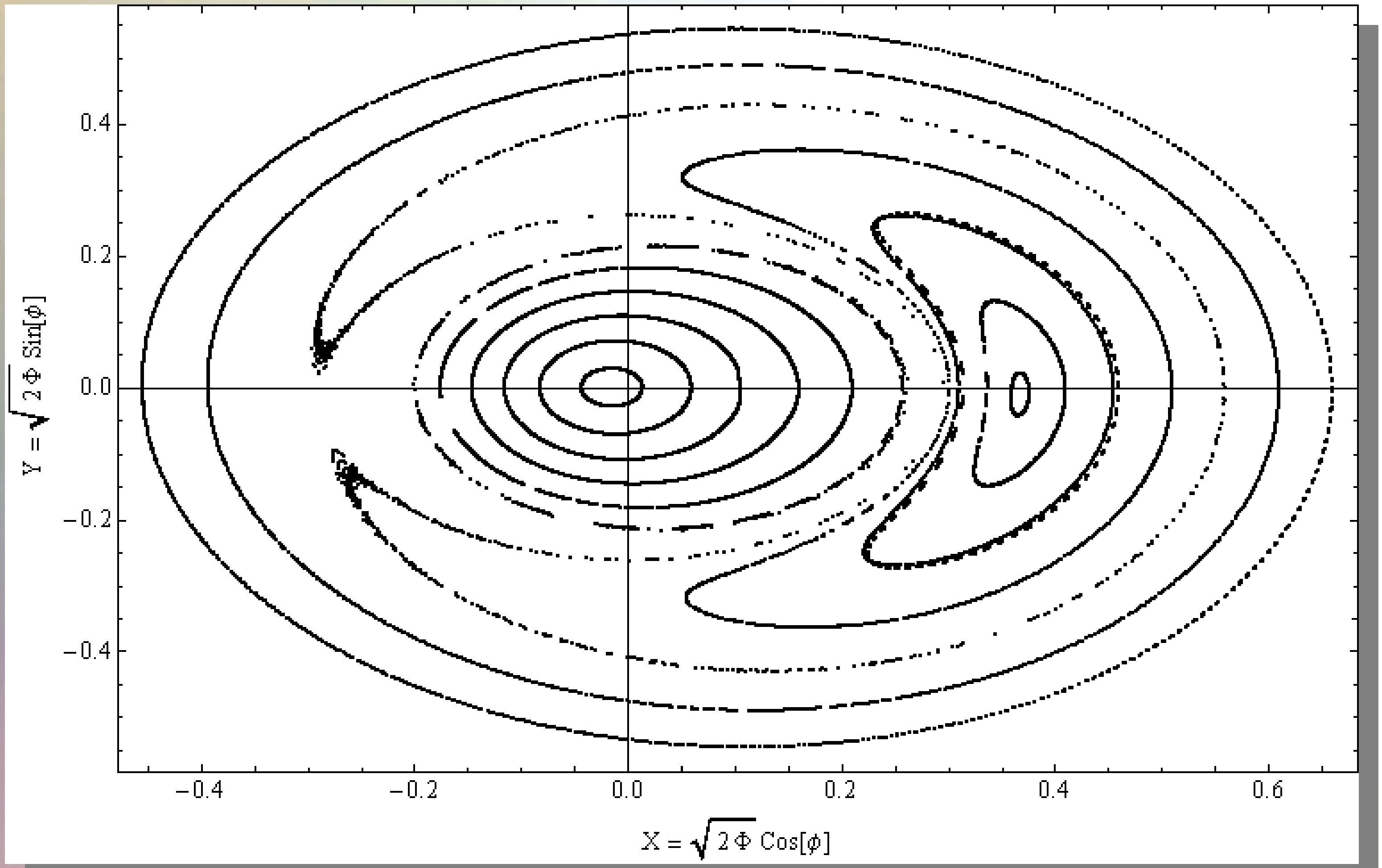
Jupiter 3:1 MMR



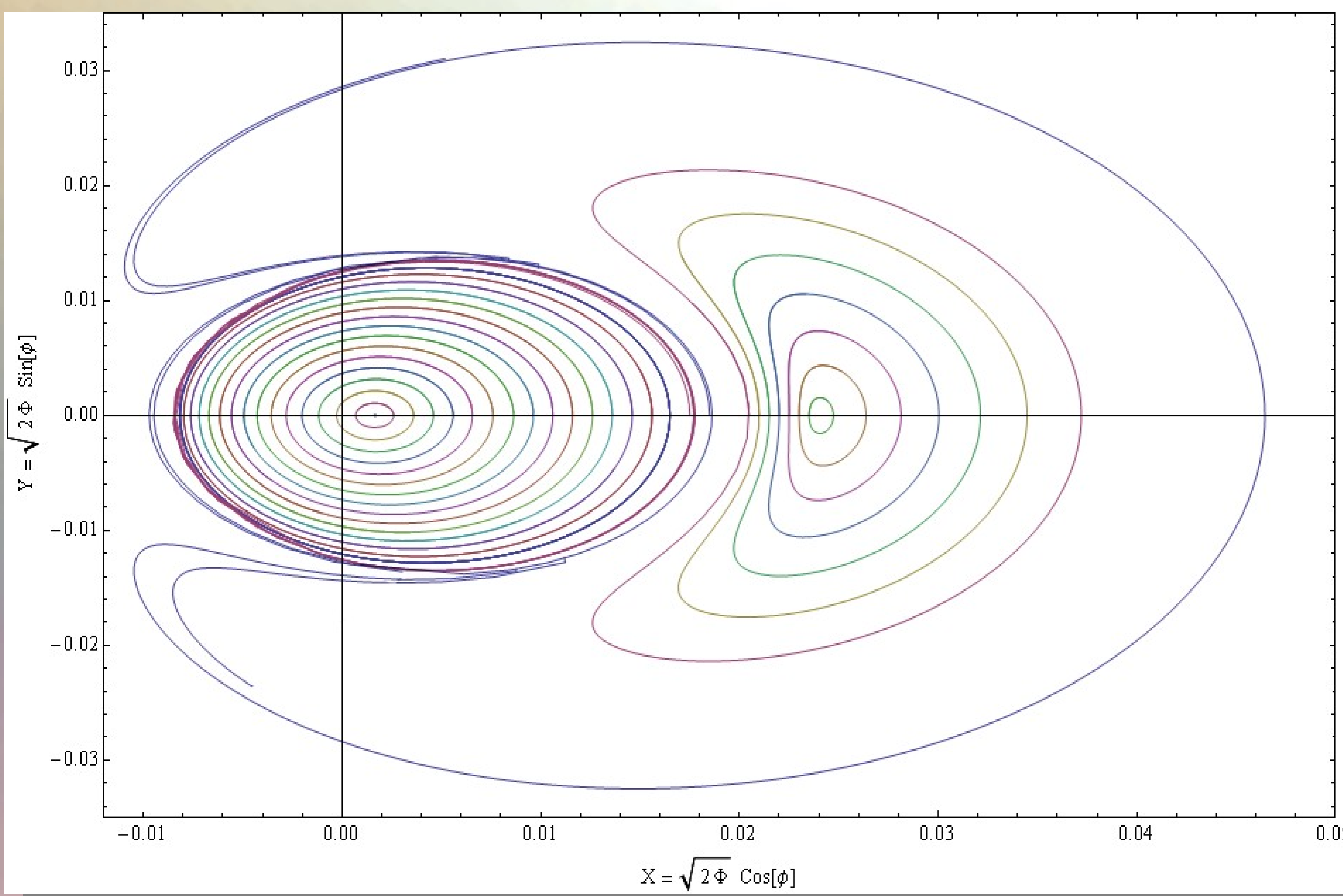


Jupiter 7:4 MMR

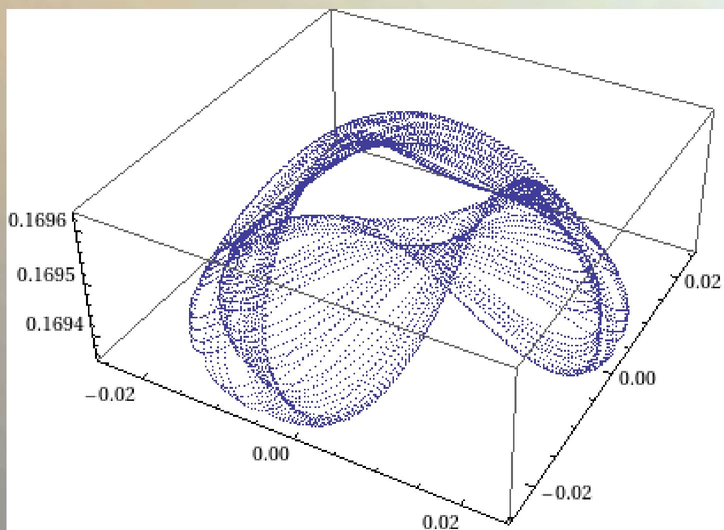
S.o.S. Jupiter 3:1 MMR



S.o.S. Venus 5:3 MMR

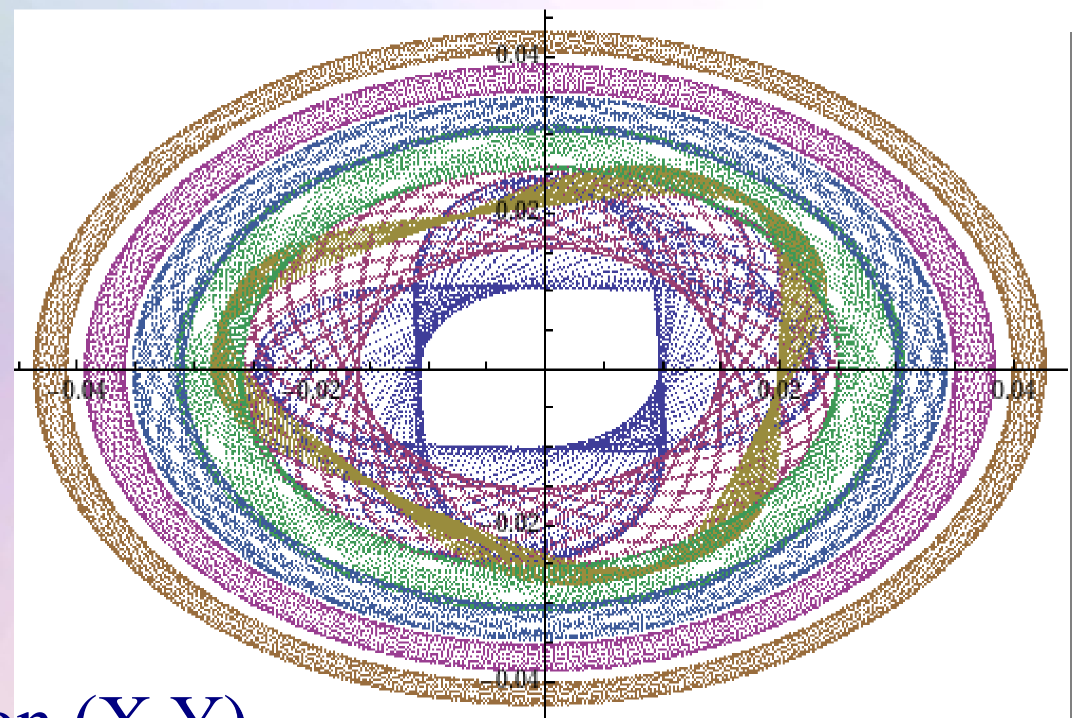


Some preliminary results...



3D projection (X,Y,J)

Neptune 2:3 outer MMR



2D projection (X,Y)