Chaos in open Hamiltonian systems

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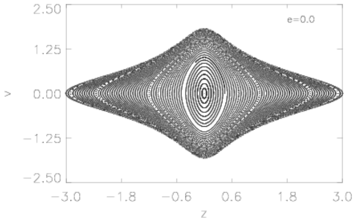
What's going on?

- Introduction - phase space of conservative dynamics
- Open Hamiltonian system
- Long-lived chaotic transients in celestial mechanics
- Outlook

Phase space structure in conservative systems

$$H(\mathbf{I}, \mathbf{\Theta}) = H_0(\mathbf{I})$$



Phase space structure in conservative systems

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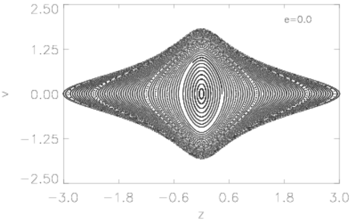
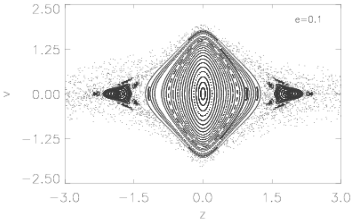


$$H(\mathbf{I}, \boldsymbol{\Theta}) = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}, \boldsymbol{\Theta})$$



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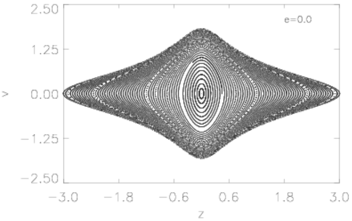
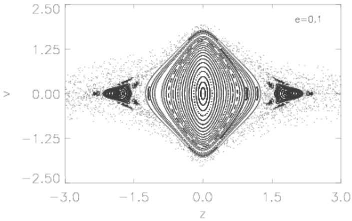
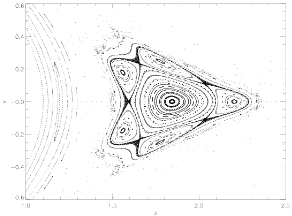
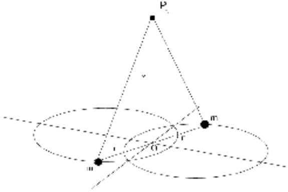






Open systems and escape

The Sitnikov problem: $H(z, \dot{z}, t) = \frac{\dot{z}}{2} - \frac{2}{\sqrt{z^2 + r(t)^2}}$



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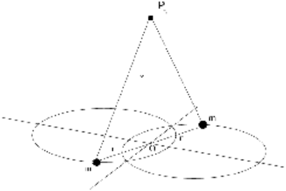

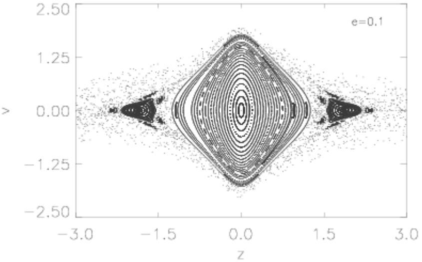

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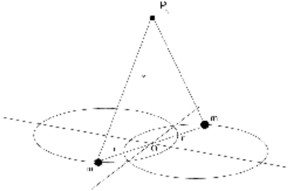

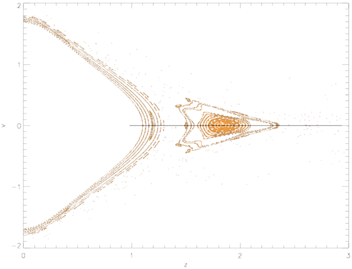

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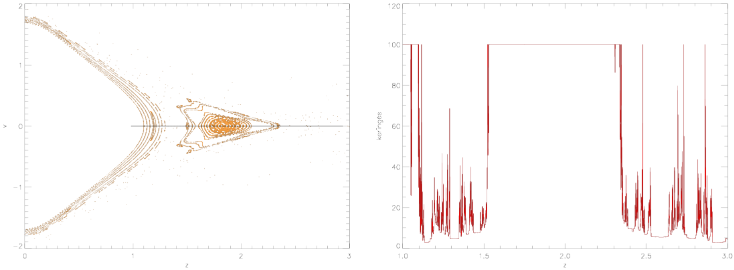

Dvorak et al. *Planetary ans Space Science* **46** 1567 (1993)

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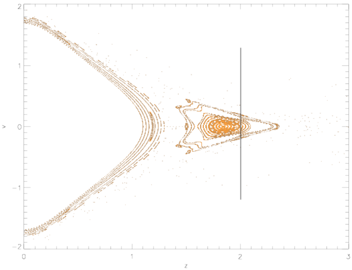

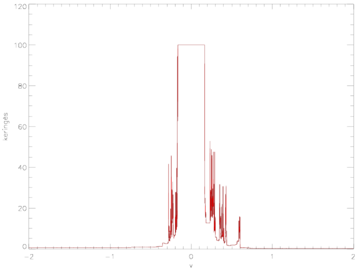

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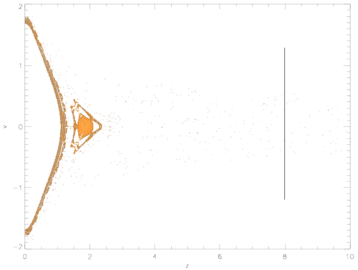

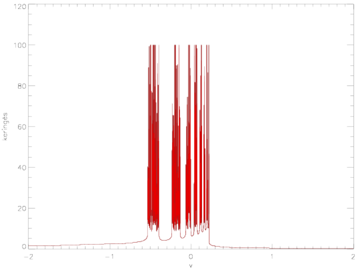

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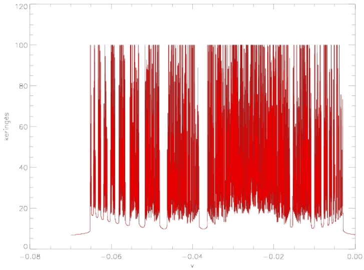

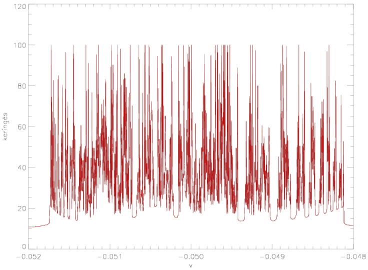

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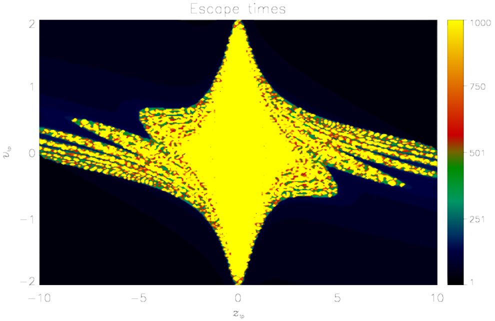
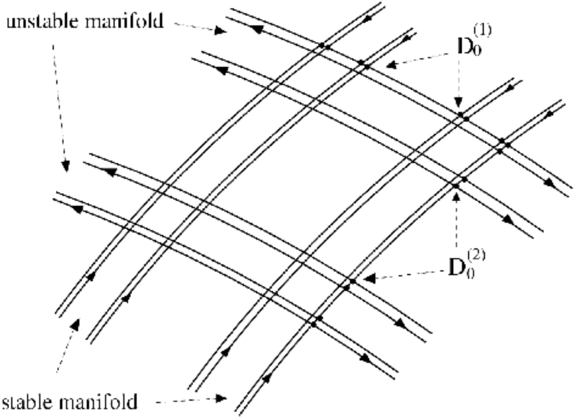


Kovács, T., Érdi, B., *Astron. Nach.* **104**, 801 (2007)

Transient chaos (chaotic scattering)



Hsu, G.-H., Ott, E. and Grebogi, C., *Phys. Lett.* **A127**, 199 (1988)

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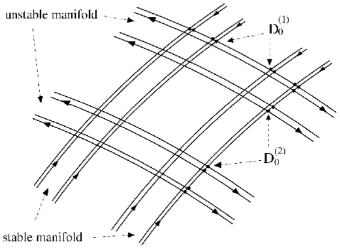


- double Cantor fractal structure
- Lebesgue measure is zero
- relation between geometry and dynamics:

$$D = 2\left(1 - \frac{\kappa}{\lambda}\right)$$

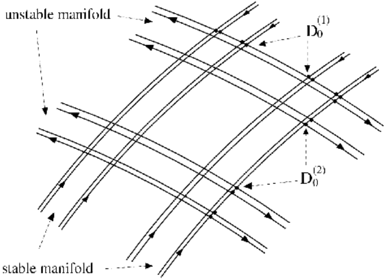
Ott, E., *Chaos in dynamical systemcs*, Cambridge Univ. Press (1993)
Kantz, H., and Grassberger, P., *Physica* **D17**, 75 (1985)

The union of all the hyperbolic periodic orbits on the saddle and of all the homoclinic and heteroclinic points formed among their manifold (or the common part of the stable and unstable manifolds of all the infinitely many periodic orbits on the saddle).

Tél, T, and Gruiz, M., *Chaotic dynamics*, Cambridge Univ. Press (2006)

Transient chaos (chaotic scattering)

Average lifetime of chaos:



We are interested in non-escaping trajectories!

$$N(t) \approx N(0)e^{-\kappa t}$$

$$\tau = \frac{1}{\kappa}$$

κ– escape rate
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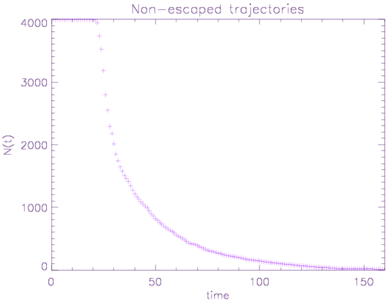

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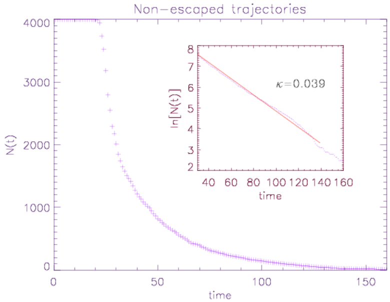

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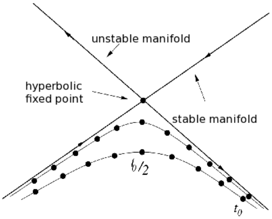

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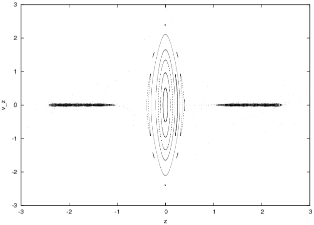

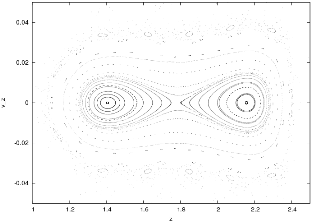

Examples in astrodynamics

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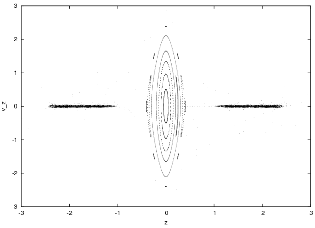

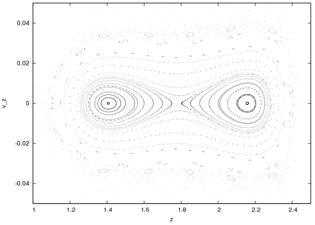

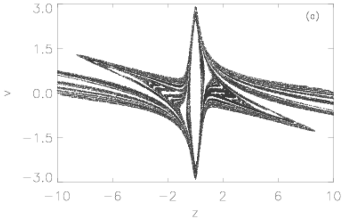

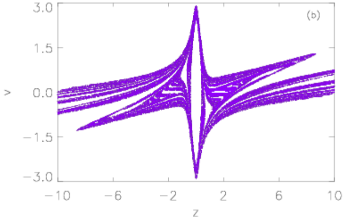

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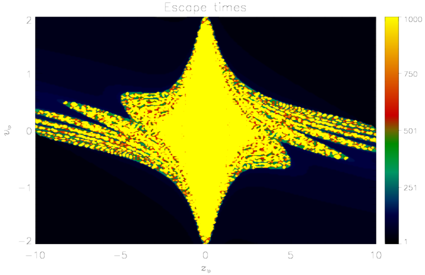

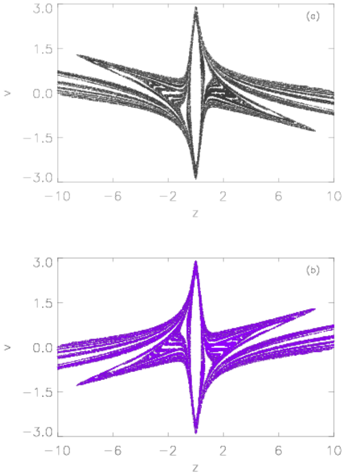

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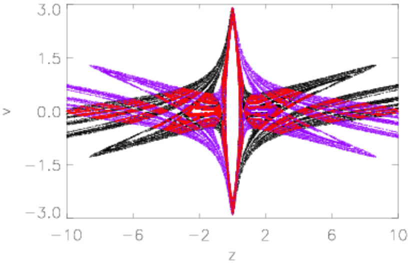

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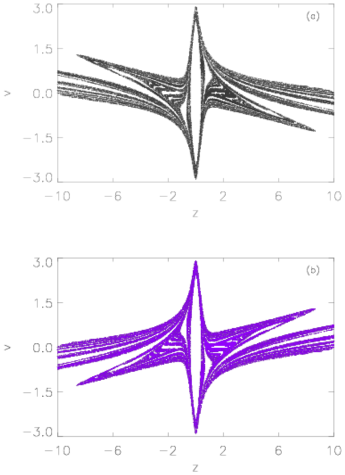
Kovács, T., Érdi, B., *Cel. Mech. and Dyn. Astron.* **104**, 237 (2009)



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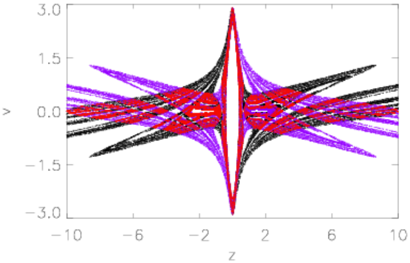

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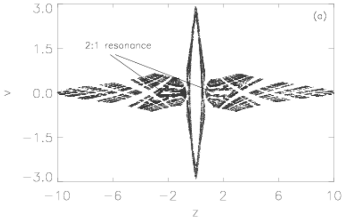
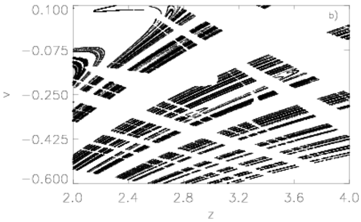
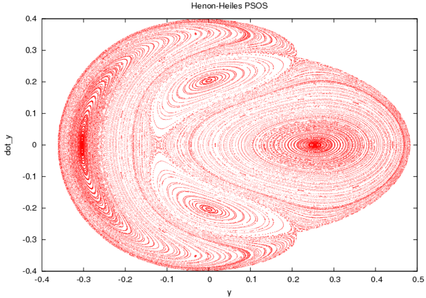




Examples in astrodynamics

Hénon-Heiles system:

$$H(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$



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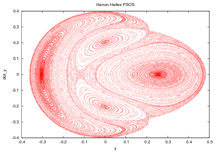

AGUIRRE, VALLEJO, AND SANJUÁN

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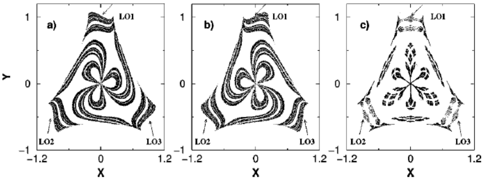

FIG. 6. Stable manifold, unstable manifold, and strange saddle for $E=0.25$. The initial conditions are (x,y) and tangential shooting, with a fine grid of 2000×2000 dots. The arrows show the three Lyapunov Orbits (LO1, LO2, and LO3).

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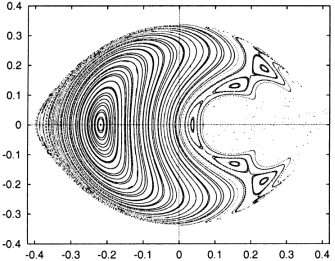
Hill's problem:

$$H(Q_1, Q_2, \dot{Q}_1, \dot{Q}_2) = \frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2 + Q_1^2 + Q_2^2) - 6(Q_1^2 + Q_2^2)(Q_1^2 - Q_2^2)^2$$



Simó, C., Stuchi, T.J., Physica **D140** 1-32 (2000)

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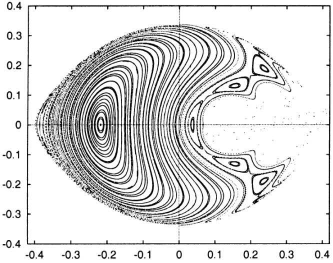

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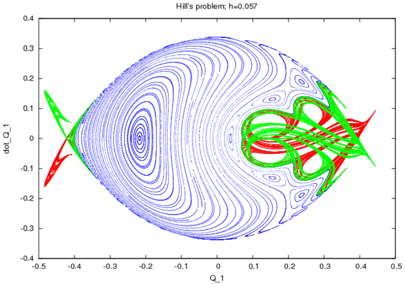

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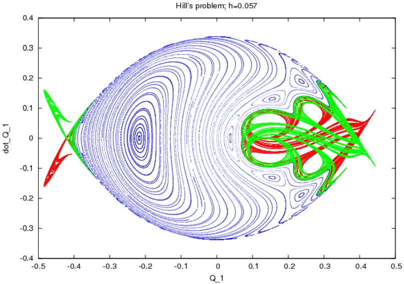

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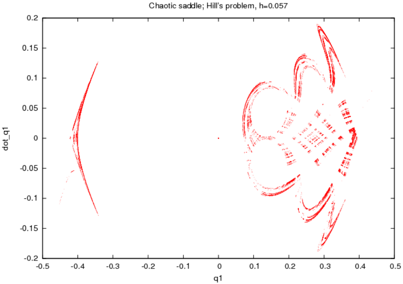

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Outlook to atomic physics

Ionization of hydrogen atom in crossed electric and magnetic field:

$$H(u, v, P_u, P_v) = \frac{1}{2}(P_u^2 + P_v^2) + \frac{1}{2}(u^2 + v^2) - \frac{\epsilon}{2\Omega}(u^4 - v^4) + \frac{1}{4\Omega^2}(u^2 + v^2)(uP_v - vP_u) + \frac{1}{32\Omega^4}(u^2 + v^2)^3$$

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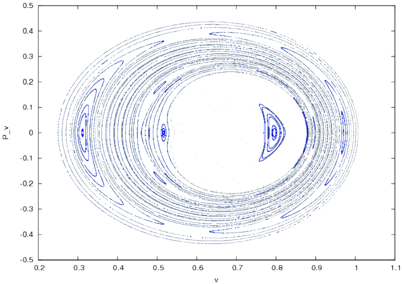

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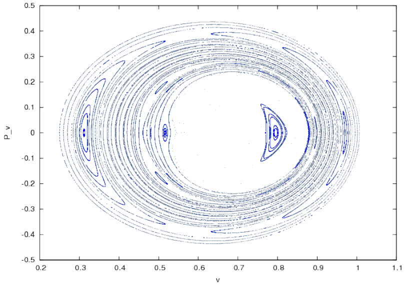

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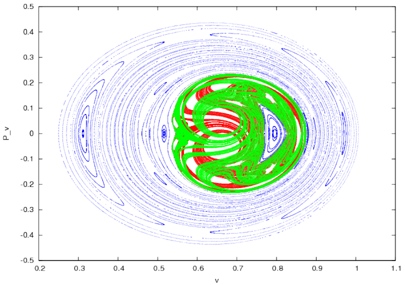

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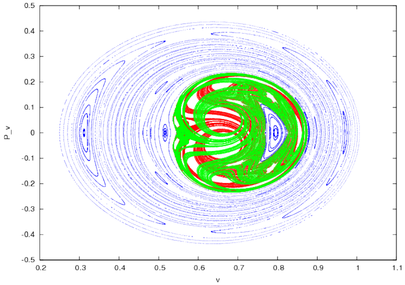

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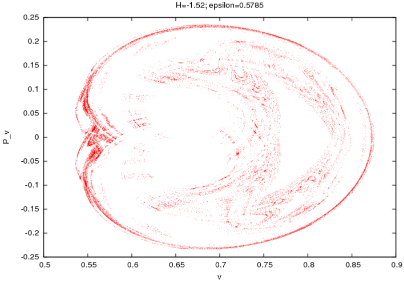

Kovács, T., unpublished

What have we learned?

- Beside the permanent chaos exist a finite time chaotic behavior - *chaotic scattering* - in open Hamiltonian systems
- The main concept is the *escape rate* i.e. the exponencial decay of the non-escaped trajectories
- There is a well-defined chaotic set - the *chaotic saddle* - in the phase space responsible for transient chaos...
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