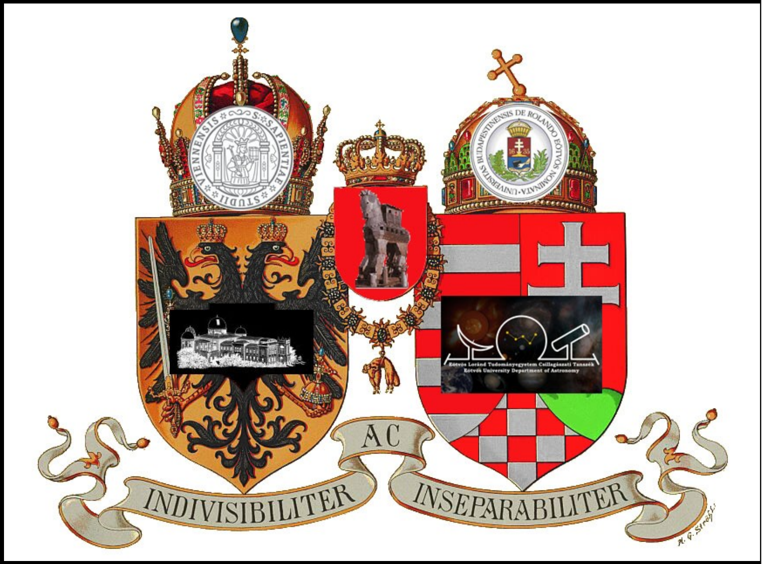
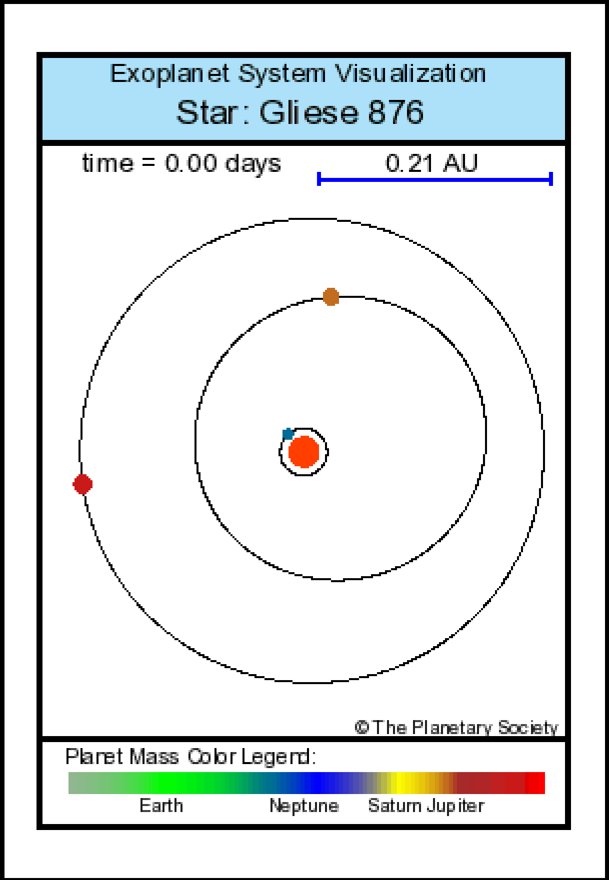
T. Borkovits[1]

Transit timing variations in hierarchical triple (exoplanetary) systems
- an analytic study

[1] Baja Astronomical Observatory, Baja

borko@alcyone.bajaobs.hu





MOTIVATION

- Increasing number of exoplanetary systems
- Lengthening time interval of the observations

⇓

- Larger sample of dynamically interesting systems (e.g. inclined axis)
- Detection of longer period systems (several months)
- Longer data series ➔ possible detection of variations (perturbations)
- More personal motivation:

  the same effects (and calculations) applied for eclipsing binaries
  in hierarchical triple stellar systems is interesting for a very few specialists only

MOTIVATION

- Increasing number of exoplanetary systems
- Lengthening time interval of the observations

⇩

- Larger sample of dynamically interesting systems (e.g. inclined axis)
- Detection of longer period systems (several months)
- Longer data series ➔ possible detection of variations (perturbations)
- More personal motivation:

  the same effects (and calculations) applied for eclipsing binaries

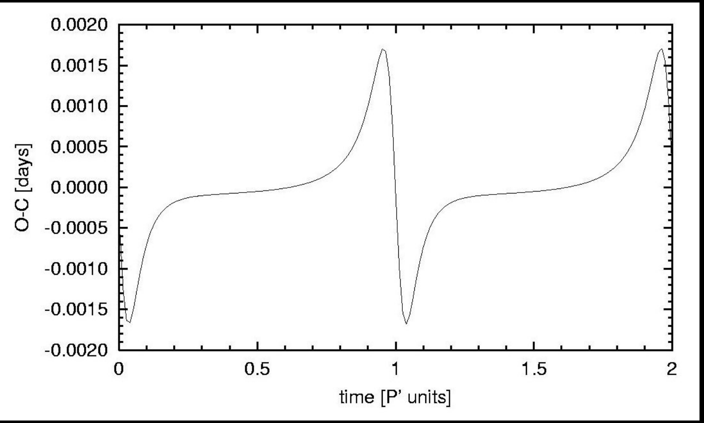
  in hierarchical triple stellar systems is interesting for a very few specialists only

I know, you unterstand what I say ☺

NEW POSSIBILITIES IN O–C ANALYSIS

Detectability of the short period perturbations

o Borkovits et al., 2003: If *P'/P*<=100*P* even the short period perturbations in the eclipsing O—C diagram (i.e. the time-delay in the occurence of the eclipsing minima) may reach 0.001 days ➔ theoretically can be detected by ground-based photometry (Perhaps IU Aur?)



The short-term perturbations in the O—C curve produced from a numeric integration of the IU Aur triple system

This valid for an eclipsing binary with typical period of a few days

But recently we know transiting exoplanets with periods of months!

NEW POSSIBILITIES IN O–C ANALYSIS

Analytical study on long-term perturbations in hierarchical triple systems with (and without) distorted components

Final purpose:

- Analytical form of the Transit Timing Variation (eclipsing O-C)!

- We must calculate the perturbations in the observational and NOT only in the dynamical frame of reference **DO NOT CONFUSE THE TWO!!!**
- Our final independent variable is the true longitude (u) measured from the intersection of the binary's orbital plane and the sky, as in the mid-eclipse moment

$$u \approx \pm \frac{\pi}{2} + 2k\pi$$

NEW POSSIBILITIES IN O–C ANALYSIS

Analytical study on long-term perturbations in hierarchical triple systems with (and without) distorted components

- For the analytical form of O–C:

Kepler-equation-"like" =>

$$\int_{t_0}^{t_N} \mathrm{d}t = \int_{\pi/2}^{2N\pi+\pi/2} \frac{a^{3/2}}{\mu^{1/2}} \frac{(1-e^2)^{3/2}}{[1+e\cos(u-\omega)]^2} \frac{\mathrm{d}u}{1-\frac{\rho_1^2}{c_1}\dot{\Omega}\cos i}$$

$$\approx \int \frac{a^{3/2}}{\mu^{1/2}} \frac{(1-e^2)^{3/2}}{[1+e\cos(u-\omega)]^2} \left(1+\frac{\rho_1^2}{c_1}\dot{\Omega}\cos i\right)\mathrm{d}u.$$

$$\Delta e \sim e\left(\frac{P}{P'}\right)^2 u,$$

$$\Delta\omega \sim \omega\left(\frac{P}{P'}\right)^2 u,$$

$$\overline{P}_I = \frac{P}{2\pi}\left[2\arctan\left(\sqrt{\frac{1-e}{1+e}}\frac{\cos\omega}{1+\sin\omega}\right)-(1-e^2)^{1/2}\frac{e\cos\omega}{1+e\sin\omega}\right]$$

$$\overline{P}_{II} = \frac{P}{2\pi}\left[2\arctan\left(\sqrt{\frac{1-e}{1+e}}\frac{-\cos\omega}{1-\sin\omega}\right)+(1-e^2)^{1/2}\frac{e\cos\omega}{1-e\sin\omega}\right]$$

$$u \approx \pm\frac{\pi}{2}+2k\pi$$

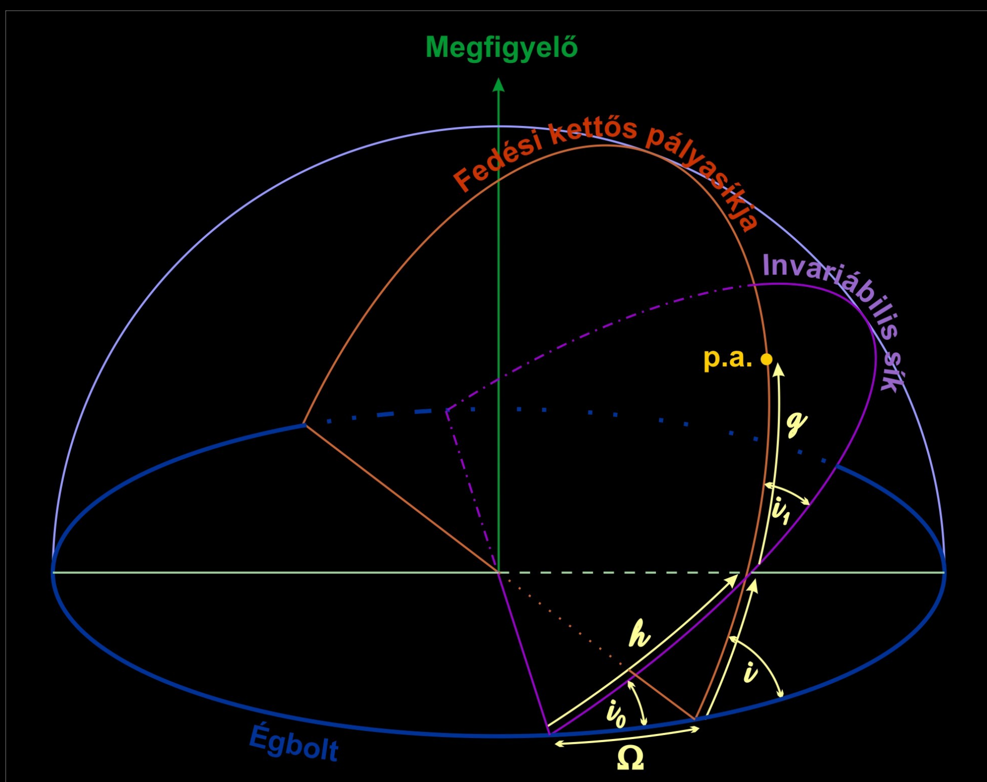
$$\overline{P}_{I,II} = P_{\mathrm{s}}E+\frac{P}{2\pi}\left[\pm\frac{1}{2}\pi \mp 2e\cos\omega+\left(\frac{3}{4}e^2+\frac{1}{8}e^4\right)\sin 2\omega \pm\left(\frac{1}{3}e^3+\frac{1}{8}e^5\right)\cos 3\omega-\frac{5}{32}e^4\sin 4\omega \mp \frac{3}{40}e^5\cos 5\omega\right]$$

PERTURBATION EQUATIONS – short-term terms

at least some of them:

$$\frac{d\overline{u}_1}{d\nu_2} \approx \frac{\mu_1^{1/2}}{a_1^{3/2}} \frac{\rho_2^2}{\sqrt{\mu_2 a_2 (1-e_2^2)}} - \frac{d\overline{\Omega}_1}{d\nu_2}\cos i_1$$

$$\approx \frac{P_2}{P_1}\frac{(1-e_2^2)^{3/2}}{(1+e_2\cos\nu_2)^2} - \frac{d\Omega_1}{d\nu_2}\cos i_1.$$



$$\frac{da_1}{d\nu_2} = 0,$$

$$\frac{de_1}{d\nu_2} = A_L e_1(1+e_2\cos\nu_2)\Big[\left(1-I^2\right)\sin 2g_1 - \frac{1}{2}(1+I)^2\sin(2\nu_2-2\nu_{2m}-2g_1) + \frac{1}{2}(1-I)^2\sin(2\nu_2-2\nu_{2m}+2g_1)\Big],$$

$$\frac{d\omega_1}{d\nu_2} = A_L(1+e_2\cos\nu_2)\left\{\frac{3}{5}\left(I^2-\frac{1}{3}\right) + \left(1-I^2\right)\left[\cos 2g_1 + \frac{3}{5}\cos(2\nu_2-2\nu_{2m})\right] + \frac{1}{2}(1+I)^2\cos(2\nu_2-2\nu_{2m}-2g_1) + \frac{1}{2}(1-I)^2\cos(2\nu_2-2\nu_{2m}+2g_1)\right\} - \frac{d\Omega_1}{d\nu_2}\cos i_1,$$

PERTURBATION EQUATIONS – short-term terms

and, finally, the direct terms are as follows:

$$\frac{\mu_1^{1/2}}{a_1^{3/2}}\left(\dot{u}_{1\mathrm{P}}\right)^{-1}_{\mathrm{dir}} = A_{\mathrm{L}}(1+e_2\cos v_2)\left\{\frac{4}{5}\left(I^2-\frac{1}{3}\right)f_1(e_1)\right.$$
$$+\frac{51}{20}\left(1-I^2\right)e_1^2 f_2(e_1)\cos 2g_1$$
$$+\frac{4}{5}\left(1-I^2\right)f_1(e_1)\cos(2v_2-2v_{2\mathrm{m}})$$
$$+\frac{51}{40}(1+I)^2 e_1^2 f_2(e_1)\cos(2v_2-2v_{2\mathrm{m}}-2g_1)$$
$$\left.+\frac{51}{40}(1-I)^2 e_1^2 f_2(e_1)\cos(2v_2-2v_{2\mathrm{m}}+2g_1)\right\}, \quad (15)$$

where

$$A_{\mathrm{L}} = \frac{15}{8}\frac{m_3}{m_{123}}\frac{P_1}{P_2}\frac{(1-e_1^2)^{1/2}}{(1-e_2^2)^{3/2}}, \quad (16)$$

and

$$f_1(e) = 1+\frac{25}{8}e^2+\frac{15}{8}e^4+\frac{95}{64}e^6, \quad (17)$$
$$f_2(e) = 1+\frac{31}{51}e^2+\frac{23}{48}e^4. \quad (18)$$

Furthermore, i_{m} denotes the mutual inclination of the two orbital planes, while

$$I = \cos i_{\mathrm{m}}, \quad (19)$$

and m_{123} stands for the total mass of the system.

and the result:

$$O-C_{P_2} = \frac{P_1}{2\pi}A_{\mathrm{L}}\left\{\left(\left(I^2-\frac{1}{3}\right)\frac{4}{5}\left(1\mp\frac{3}{2}e_1\sin\omega_1\right)\mathcal{M}\right.\right.$$
$$+\left(1-I^2\right)\frac{2}{5}\left(1\mp\frac{3}{2}e_1\sin\omega_1\right)\mathcal{S}(2v_2-2v_{2\mathrm{m}})$$
$$\mp\left(1-I^2\right)2e_1\sin(\omega_1-2g_1)\mathcal{M}$$
$$\pm(1+I)^2\frac{1}{2}e_1\mathcal{C}(2v_2-2v_{2\mathrm{m}}+\omega_1-2g_1)$$
$$\mp(1-I)^2\frac{1}{2}e_1\mathcal{C}(2v_1-2v_{2\mathrm{m}}-\omega_1+2g_1)$$
$$+\cot i_1\sin i_{\mathrm{m}}\left[-I\frac{2}{5}(1\mp 2e_1\sin\omega_1)\cos u_{1\mathrm{m}}\mathcal{M}\right.$$
$$+(1+I)\frac{1}{10}(1\mp 2e_1\sin\omega_1)\mathcal{S}(2v_2-2v_{2\mathrm{m}}+u_{1\mathrm{m}})$$
$$\left.\left.-(1-I)\frac{1}{10}(1\mp 2e_1\sin\omega_1)\mathcal{S}(2v_2-2v_{2\mathrm{m}}-u_{1\mathrm{m}})\right]\right\}$$
$$+\mathcal{O}(e_1^2), \quad (20)$$

where

$$\mathcal{M} = \int 1+e_2\cos v_2 dv_2$$
$$= v_2-l_2+e_2\sin v_2 \quad (21)$$
$$= 3e_2\sin v_2-\frac{3}{4}e_2^2\sin 2v_2+\frac{1}{3}e_2^3\sin 3e_2+\mathcal{O}(e_2^4), \quad (22)$$

furthermore,

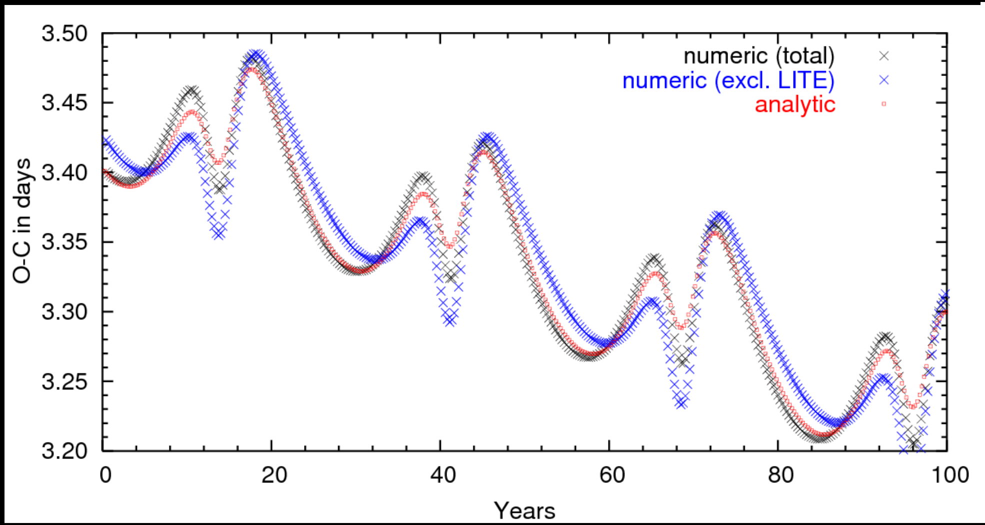
$$\mathcal{S}(x) = \sin(x)+e_2\sin(x-v_2)+\frac{1}{3}e_2\sin(x+v_2), \quad (23)$$
$$\mathcal{C}(x) = \cos(x)+e_2\cos(x-v_2)+\frac{1}{3}e_2\cos(x+v_2). \quad (24)$$

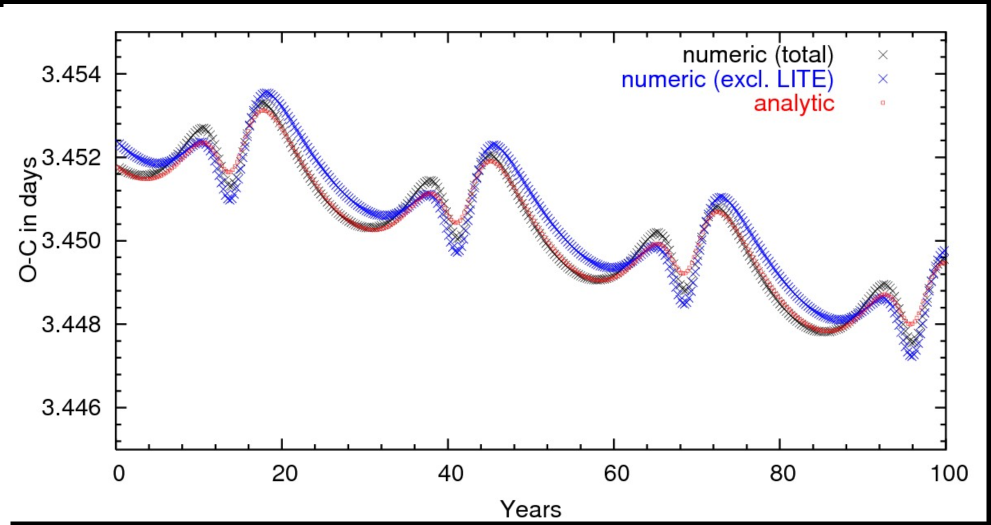
DISCUSSION – with illustration on COROT-9B

Combination of short-period dynamic and pure geometric (LITE) effects on the O–C curve of CoRoT-9b

$m_{star}=0.99\ m_{Sun}$ $m_{planet}=0.0008\ m_{Sun}$

$P_1=95^d$ $P_2=10{,}000^d$





$m_3=1.0\ m_{Sun}$ $m_3=0.01\ m_{Sun}$

Physical and orbital parameter of CoRoT-9b and its host star from Deeg et al. 2010, Nature

PERTURBATION EQUATIONS – long-term terms

at least some of them:

$$\frac{du_1}{dg_1} = \frac{1}{A + B\cos 2g_1} - \frac{d\Omega_1}{dg_1}\cos i_1,$$

$$\frac{da_1}{dg_1} = 0,$$

$$\frac{1}{e_1}\frac{de_1}{dg_1} = \frac{A_1 \sin 2g}{A + B\cos 2g},$$

$$\frac{dh_1}{dg_1} = -\frac{1}{\cos j_1}\frac{A_{n1} - A_{n2}\cos 2g}{A + B\cos 2g},$$

$$\cot j_1 \frac{dj_1}{dg_1} = \frac{-A_{n2}\sin 2g}{A + B\cos 2g},$$

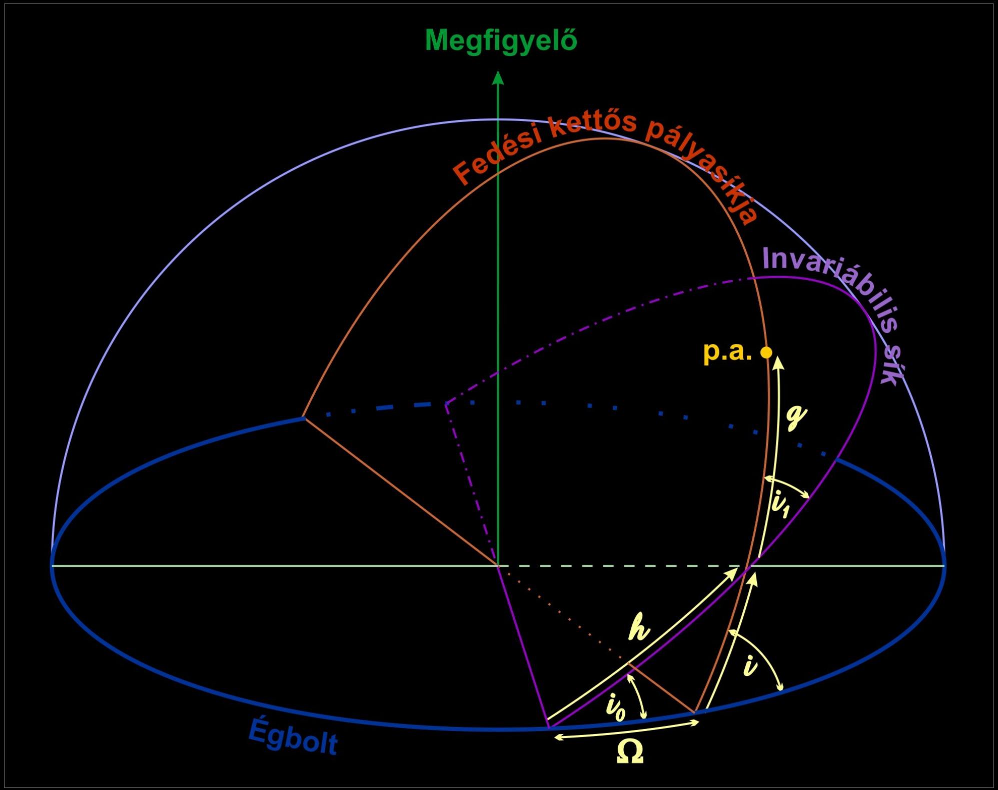
$$\frac{d\omega_1}{dg_1} = 1 + \frac{dh_1}{dg_1}\cos j_1 - \frac{d\Omega_1}{dg_1}\cos i_1,$$

$$\frac{d\Omega_1}{dg_1} = \frac{dh_1}{dg_1}\frac{\sin j_1}{\sin i_1}\cos u_m + \frac{dj_1}{dg_1}\frac{1}{\sin i_1}\sin u_m,$$

$$\frac{di_1}{dg_1} = -\frac{dh_1}{dg_1}\sin j_1 \sin u_m + \frac{dj_1}{dg_1}\cos u_m,$$

while for the direct term

$$\frac{\mu_1^{1/2}}{a_1^{3/2}}(\dot{u}_1)^{-1}_{dir} = \frac{A_d + B_d\cos 2g}{A + B\cos 2g},$$



where

$$A = A_G(1 - e_1^2)^{-1}\left[I^2 - \frac{1}{5}(1 - e_1^2) + \frac{2}{5}\left(1 + \frac{3}{2}e_1^2\right)\frac{G_1}{G_2}I\right],$$

$$B = A_G(1 - e_1^2)^{-1}\left[1 - e_1^2 - I^2 - e_1^2\frac{G_1}{G_2}I\right],$$

PERTURBATION EQUATIONS – long-term terms

Solutions:

Restrictions:

- Second order perturbing function (in a_1/a_2 ratio)
- Very strong hierarchicity (i. e. G_1/G_2 is negligible)
- Point masses

The Hamiltonian does not depend on neither any of the elements of the third companion, nor H_1 .

$$(1 - 5\cos 2g)(\eta - 1)\left(\eta - \theta^2\right) + 4\frac{\theta^2}{\eta_{1/5}}\left(\eta_{1/5} - \eta\right) = 0,$$

where

$$\eta = 1 - e^2,$$

$$\theta = \frac{H_1}{L_1}.$$

PERTURBATION EQUATIONS – long-term terms

Solutions:

Analytical solution of:

$$\frac{d\zeta}{du} = \mp A_0 \frac{2}{5} \sqrt{6} \sqrt{4\zeta^3 - g_2\zeta - g_3},$$

↓

is Weierstrass elliptic funtion.

where:

$$\zeta = \eta - \frac{1}{9}\left(5 + 5\theta^2 + \eta_0\right)$$

$$g_2 = \frac{20}{3}\left[\left(\frac{5}{9}\left(1+\theta^2\right) - \frac{7}{9}\eta_0\right)\left(1+\theta^2\right) + \frac{19}{45}\eta_0^2 - \theta^2\right],$$

$$g_3 = \frac{1000}{729}\left(1+\theta^2\right)^3 - \frac{700}{243}\eta_0\left(1+\theta^2\right)^2 + \frac{20}{27}\left(\frac{11}{9}\eta_0^2 - 5\theta^2\right)\left(1+\theta^2\right) + \frac{32}{27}\left(5\theta^2 + \frac{7}{27}\eta_0^2\right)\eta_0$$

(See Kozai 1962, Söderhjelm 1982 for details)

Not really practical for our purpose

PERTURBATION EQUATIONS – long-term terms

Solutions:

Approximative solution for some centuries:

$$\eta = \frac{1}{2}\left(1+\theta^2\right) + \frac{1}{2}\frac{Z}{x} - \frac{1}{2x}\sqrt{\left[Z-(X+Y)\,x\right]^2 - 4XYx(x-x_0)},$$

where:

$$X = \eta_{g0} - 1,$$
$$Y = \eta_{g0} - \theta^2,$$
$$Z = \frac{4}{5}\frac{\theta^2}{\eta_{1/5}} = \frac{XY}{\eta_{g0}-\eta_{1/5}}x_0$$

$$\eta^{1/2} = \eta_{g0}^{1/2} + \frac{1}{2}\eta_{g0}^{-1/2}\frac{XY}{B(x_0)}(x-x_0) + \frac{1}{8}\eta_{g0}^{-3/2}\frac{XY}{B(x_0)}\frac{4\eta_{g0}XYx_0 + \left[4\eta_{g0}(X+Y)-XY\right]B(x_0)}{B(x_0)^2}(x-x_0)^2 + \ldots,$$

$$e = e_{g0} - \frac{1}{2}e_{g0}^{-1}\frac{XY}{B(x_0)}(x-x_0) - \frac{1}{8}e_{g0}^{-3}\frac{XY}{B(x_0)}\frac{4e_{g0}^2XYx_0 + \left[4e_{g0}^2(X+Y)+XY\right]B(x_0)}{B(x_0)^2}(x-x_0)^2 + \ldots,$$

$$A_G\frac{\mathrm{d}u}{\mathrm{d}g} = \frac{\eta_{g0}^{1/2}}{B(x_0)} + \frac{1}{2}\eta_{g0}^{-1/2}\frac{4\eta_{g0}XYx_0 + \left[XY+2\eta_{g0}(X+Y)\right]B(x_0)}{B(x_0)^3}(x-x_0)$$
$$+\frac{1}{8}\eta_{g0}^{-3/2}\left\{\frac{48\eta_{g0}^2X^2Y^2x_0^2 + \left[48\eta_{g0}^2(X+Y)XYx_0 + 12\eta_{g0}X^2Y^2x_0\right]B(x_0)}{B(x_0)^5}\right.$$
$$\left.+\frac{\left[8\eta_{g0}^2(X+Y)^2 + 16\eta_{g0}^2XY - X^2Y^2 + 8XY\eta_{g0}(X+Y)\right]B(x_0)^2}{B(x_0)^5}\right\}(x-x_0)^2$$

Note: $\left(\frac{\mathrm{d}u}{\mathrm{d}g}\right)_0 = A_G^{-1}\frac{\eta_{g0}^{3/2}}{\theta^2 - \frac{1}{5}\eta_{g0}^2 + \left(\eta_{g0}^2-\theta^2\right)\cos 2g_0}$ simply

PERTURBATION EQUATIONS – long-term terms

Solutions:

Approximative solution for some centuries:

$$u - u_0 = \left(\mu_0 + \mu_2 + \mu_1\cos 2g_0 + \frac{1}{2}\mu_2\cos 4g_0\right)(g - g_0) - \frac{1}{2}(\mu_1 + 2\mu_2\cos 2g_0)(\sin 2g - \sin 2g_0) + \frac{1}{8}\mu_2(\sin 4g - \sin 4g_0)$$

$$\mathcal{G} = \Pi u = g + \gamma_2 \sin 2g + \gamma_4 \sin 4g + \ldots,$$

where:

$$\Pi^{-1} = \mu_0 + \mu_2 + \mu_1\cos 2g_0 + \frac{1}{2}\mu_2\cos 4g_0 + \ldots$$

$$= \frac{1}{A_G}\frac{\eta_{g0}^{1/2}}{B(x_0)}\left\{1 + \frac{1}{2}\eta_{g0}^{-1}\frac{4\eta_{g0}XYx_0 + \left[XY + 2\eta_{g0}(X+Y)\right]B(x_0)}{B(x_0)^2}\cos 2g_0\right.$$

$$+\frac{1}{8}\eta_{g0}^{-2}\left\{\frac{48\eta_{g0}^2X^2Y^2x_0^2 + 12\eta_{g0}XYx_0\left[4\eta_{g0}(X+Y) + XY\right]B(x_0)}{B(x_0)^4}\right.$$

$$\left.\left.+\frac{\left[8\eta_{g0}^2(X+Y)^2 + 16\eta_{g0}^2XY - X^2Y^2 + 8XY\eta_{g0}(X+Y)\right]B(x_0)^2}{B(x_0)^4}\right\}\left(1 + \frac{1}{2}\cos 4g_0\right)\right\} + \ldots,$$

$$\gamma_2 = -\frac{1}{2}\Pi^{-1}(\mu_1 + 2\mu_2\cos 2g_0),$$

$$\gamma_4 = \frac{1}{8}\Pi^{-1}\mu_2.$$

$$g = \mathcal{G} + G_2\sin 2\mathcal{G} + G_4\sin 4\mathcal{G} + \ldots,$$

where:

$$G_n = \frac{1}{\pi}\int_0^{2\pi}(g - \mathcal{G})\sin n\mathcal{G}(g)\frac{d\mathcal{G}}{dg}dg$$

PERTURBATION EQUATIONS – long-term terms, distorted components

Solutions:

Restrictions:

- Second order perturbing function (in a_1/a_2 ratio)
- Radial tidal forces in the close binary, mass-point ternary
- Constant angular velocity vectors

As far as the orbital elements on the r.h.s. of perturbation eqs. can be treated as constants, there closed form solutions.

$$g = \arctan\left[\sqrt{\frac{1+E_0}{1-E_0}}\tan\left(W_0+\Pi_0 u\right)\right],$$

$$e = e_0 + \frac{1}{2}e_0\frac{A_{\mathrm{r1t}}}{B}\ln\left[\frac{1-E_0\cos(2W_0+2\Pi_0 u)}{1-E_0\cos 2W_0}\right],$$

$$h = h_0 + 2\pi\chi_0 u + \kappa_0\left\{\arctan\left[\sqrt{\frac{1+E_0}{1-E_0}}\tan\left(W_0+\Pi_0 u\right)\right] - \arctan\left[\sqrt{\frac{1+E_0}{1-E_0}}\tan W_0\right]\right\},$$

$$u - u_0 = \frac{1}{2\pi}\frac{1}{A_0\sqrt{1-E_0^2}}(W-W_0)$$

PERTURBATION EQUATIONS – long-term terms, distorted components

Solutions:

- For large mutual inclination the eccentricity cannot be treated as constant. At a relatively close system with medium eccentricity the convergence is very weak, so we need to calculate for higher orders. E.g. here I give the form of the angular velocity of the apsidal motion up to fifth order in e:

$$\Pi^* = \Pi\Bigg\{1+\left[\frac{1}{2}\frac{F_0}{1-E_0^2}(1-N_0)+\frac{1}{8}\frac{F_0}{1-E_0^2}X_1-\frac{1}{8}\frac{G_0}{1-E_0^2}-\frac{1}{4}\frac{F_0^2}{(1-E_0^2)^2}-\frac{1}{4}Y_1(1-N_0)-\frac{1}{8}Y_1X_1-\frac{1}{8}Y_2+\frac{1}{8}\frac{F_0}{1-E_0^2}Y_1+\frac{1}{8}Y_1^2\right]\mathcal{E}_0^2+$$

$$+\left[\frac{7}{32}\frac{F_0}{1-E_0^2}(1-N_0)^3+\frac{19}{128}\frac{F_0}{1-E_0^2}X_1(1-N_0)^2+\frac{3}{64}\frac{F_0}{1-E_0^2}X_1^2(1-N_0)+\frac{3}{512}\frac{F_0}{1-E_0^2}X_1^3-\frac{19}{128}\frac{G_0}{1-E_0^2}(1-N_0)^2-\right.$$

$$-\frac{9}{64}\frac{G_0}{1-E_0^2}X_1(1-N_0)-\frac{21}{512}\frac{G_0}{1-E_0^2}X_1^2-\frac{45}{128}\frac{F_0^2}{(1-E_0^2)^2}(1-N_0)^2-\frac{49}{128}\frac{F_0^2}{(1-E_0^2)^2}X_1(1-N_0)-\frac{21}{512}\frac{F_0^2}{(1-E_0^2)^2}X_1^2-$$

$$-\frac{1}{64}\frac{F_0G_0}{(1-E_0^2)^2}(1-N_0)+\frac{3}{128}\frac{F_0G_0}{(1-E_0^2)^2}X_1-\frac{3}{512}\frac{G_0^2}{(1-E_0^2)^2}-\frac{5}{32}Y_1(1-N_0)^3-\frac{13}{128}X_1Y_1(1-N_0)^2-\frac{3}{128}X_1^2Y_1(1-N_0)-$$

$$\left.-\frac{3}{512}X_1^3Y_1-\frac{13}{128}Y_2(1-N_0)^2-\frac{9}{128}X_1Y_2(1-N_0)-\frac{21}{512}X_1^2Y_2\right]\mathcal{E}_0^4+$$

$$+\Bigg\{\frac{1}{2}\frac{F_0}{1-E_0^2}-\frac{1}{2}Y_1+\left[\frac{1}{8}\frac{F_0}{1-E_0^2}(1-N_0)^2+\frac{7}{32}\frac{F_0}{1-E_0^2}X_1(1-N_0)+\frac{3}{64}\frac{F_0}{1-E_0^2}X_1^2-\frac{7}{32}\frac{G_0}{1-E_0^2}(1-N_0)-\frac{9}{64}\frac{G_0}{1-E_0^2}X_1-\right.$$

$$-\frac{1}{2}\frac{F_0^2}{(1-E_0^2)^2}(1-N_0)-\frac{5}{16}\frac{F_0^2}{(1-E_0^2)^2}X_1+\frac{1}{32}\frac{F_0G_0}{(1-E_0^2)^2}-\frac{3}{64}X_1^2Y_1-\frac{1}{8}Y_1(1-N_0)^2-\frac{3}{32}X_1Y_1(1-N_0)-\frac{3}{32}Y_2(1-N_0)-$$

$$\left.-\frac{9}{64}X_1Y_2-\frac{3}{32}\frac{F_0^3}{(1-E_0^2)^3}-\frac{1}{16}\frac{F_0}{1-E_0^2}Y_1(1-N_0)+\frac{5}{32}\frac{F_0}{1-E_0^2}X_1Y_1-\frac{1}{64}\frac{F_0}{1-E_0^2}Y_2+\frac{7}{64}\frac{G_0}{1-E_0^2}Y_1\right]\mathcal{E}_0^2+$$

$$+\left[\frac{23}{192}\frac{F_0}{1-E_0^2}X_1(1-N_0)^3+\frac{3}{128}\frac{F_0}{1-E_0^2}X_1^2(1-N_0)^2+\frac{5}{384}\frac{F_0}{1-E_0^2}X_1^3(1-N_0)-\frac{7}{48}\frac{G_0}{1-E_0^2}(1-N_0)^3-\right.$$

$$\left.-\frac{49}{256}\frac{G_0}{1-E_0^2}X_1(1-N_0)^2-\frac{77}{768}\frac{G_0}{1-E_0^2}X_1^2(1-N_0)-\frac{25}{1024}\frac{G_0}{1-E_0^2}X_1^3\right]\mathcal{E}_0^4\Bigg\}\mathcal{E}_0\cos 2g_0+$$

$$+\Bigg\{-\frac{1}{8}\frac{F_0}{1-E_0^2}(1-N_0)+\frac{1}{16}\frac{F_0}{1-E_0^2}X_1-\frac{1}{16}\frac{G_0}{1-E_0^2}-\frac{1}{16}\frac{F_0^2}{(1-E_0^2)^2}+\frac{1}{8}Y_1(1-N_0)-\frac{1}{16}X_1Y_1-\frac{1}{16}Y_2-$$

$$-\frac{1}{16}\frac{F_0}{1-E_0^2}Y_1+\frac{1}{8}Y_1^2+\left[-\frac{1}{16}\frac{F_0}{1-E_0^2}(1-N_0)^3-\frac{1}{48}\frac{F_0}{1-E_0^2}X_1(1-N_0)^2+\frac{1}{64}\frac{F_0}{1-E_0^2}X_1^2(1-N_0)^2+\frac{1}{192}\frac{F_0}{1-E_0^2}X_1^3+\right.$$

$$+\frac{1}{48}\frac{G_0}{1-E_0^2}(1-N_0)^2-\frac{3}{64}\frac{G_0}{1-E_0^2}X_1(1-N_0)-\frac{7}{192}\frac{G_0}{1-E_0^2}X_1^2+\frac{3}{32}\frac{F_0^2}{(1-E_0^2)^2}(1-N_0)^2-\frac{1}{16}\frac{F_0^2}{(1-E_0^2)^2}X_1(1-N_0)-$$

$$-\frac{1}{32}\frac{F_0^2}{(1-E_0^2)^2}X_1^2-\frac{7}{96}\frac{F_0G_0}{(1-E_0^2)^2}(1-N_0)+\frac{1}{192}\frac{F_0G_0}{(1-E_0^2)^2}X_1-\frac{1}{192}\frac{G_0^2}{(1-E_0^2)^2}+\frac{1}{16}Y_1(1-N_0)^3-\frac{1}{96}X_1Y_1(1-N_0)^2-$$

$$\left.-\frac{1}{192}X_1^3Y_1-\frac{1}{96}Y_2(1-N_0)^2-\frac{7}{192}X_1^2Y_2\right]\mathcal{E}_0^2\Bigg\}\mathcal{E}_0^2\cos 4g_0+$$

$$+\Bigg\{\frac{1}{24}\frac{F_0}{1-E_0^2}(1-N_0)^2-\frac{1}{32}\frac{F_0}{1-E_0^2}X_1(1-N_0)+\frac{1}{192}\frac{F_0}{1-E_0^2}X_1^2+\frac{1}{32}\frac{G_0}{1-E_0^2}(1-N_0)-\frac{1}{64}\frac{G_0}{1-E_0^2}X_1+$$

$$+\frac{1}{24}\frac{F_0^2}{(1-E_0^2)^2}(1-N_0)-\frac{1}{48}\frac{F_0^2}{(1-E_0^2)^2}X_1-\frac{1}{96}\frac{F_0G_0}{(1-E_0^2)^2}-\frac{1}{24}Y_1(1-N_0)^2+\frac{1}{32}X_1Y_1(1-N_0)-\frac{1}{192}X_1^2Y_1+$$

$$+\frac{1}{32}Y_2(1-N_0)-\frac{1}{64}X_1Y_2-\frac{1}{32}\frac{F_0^3}{(1-E_0^2)^3}+\frac{1}{48}\frac{F_0}{1-E_0^2}Y_1(1-N_0)-\frac{1}{96}\frac{F_0}{1-E_0^2}X_1Y_1-\frac{1}{64}\frac{F_0}{1-E_0^2}Y_2+\frac{5}{192}\frac{G_0}{1-E_0^2}Y_1+$$

$$+\left[-\frac{13}{384}\frac{F_0}{1-E_0^2}X_1(1-N_0)^3-\frac{1}{64}\frac{F_0}{1-E_0^2}X_1^2(1-N_0)^2-\frac{5}{1536}\frac{F_0}{1-E_0^2}X_1^3(1-N_0)+\frac{5}{768}\frac{G_0}{1-E_0^2}(1-N_0)^3+\right.$$

$$\left.+\frac{17}{1024}\frac{G_0}{1-E_0^2}X_1(1-N_0)^2+\frac{7}{3072}\frac{G_0}{1-E_0^2}X_1^2(1-N_0)-\frac{25}{4096}\frac{G_0}{1-E_0^2}X_1^3\right]\mathcal{E}_0^2\Bigg\}\mathcal{E}_0^3\cos 6g_0+$$

$$+\left[-\frac{1}{64}\frac{F_0}{1-E_0^2}(1-N_0)^3+\frac{11}{768}\frac{F_0}{1-E_0^2}X_1(1-N_0)^2-\frac{1}{256}\frac{F_0}{1-E_0^2}X_1^2(1-N_0)+\frac{1}{3072}\frac{F_0}{1-E_0^2}X_1^3-\frac{11}{768}\frac{G_0}{1-E_0^2}(1-N_0)^2+\right.$$

$$+\frac{3}{256}\frac{G_0}{1-E_0^2}X_1(1-N_0)-\frac{7}{3072}\frac{G_0}{1-E_0^2}X_1^2-\frac{1}{64}\frac{F_0^2}{(1-E_0^2)^2}(1-N_0)^2+\frac{1}{32}\frac{F_0^2}{(1-E_0^2)^2}X_1(1-N_0)+\frac{1}{768}\frac{F_0G_0}{(1-E_0^2)^2}(1-N_0)$$

$$-\frac{7}{1536}\frac{F_0G_0}{(1-E_0^2)^2}X_1+\frac{1}{1536}\frac{G_0^2}{(1-E_0^2)^2}+\frac{1}{64}Y_1(1-N_0)^3-\frac{11}{768}X_1Y_1(1-N_0)^2+\frac{1}{256}X_1^2Y_1(1-N_0)-$$

$$\left.-\frac{1}{3072}X_1^3Y_1-\frac{11}{768}Y_2(1-N_0)^2+\frac{3}{256}X_1Y_2(1-N_0)-\frac{7}{3072}X_1^2Y_2\right]\mathcal{E}_0^4\cos 8g_0+$$

$$+\left[-\frac{5}{384}\frac{F_0}{1-E_0^2}X_1(1-N_0)^3-\frac{1}{1536}\frac{F_0}{1-E_0^2}X_1^3(1-N_0)+\frac{5}{768}\frac{G_0}{1-E_0^2}(1-N_0)^3-\frac{7}{1024}\frac{G_0}{1-E_0^2}X_1(1-N_0)^2+\right.$$

$$\left.+\frac{7}{3072}\frac{G_0}{1-E_0^2}X_1^2(1-N_0)-\frac{1}{4096}\frac{G_0}{1-E_0^2}X_1^3\right]\mathcal{E}_0^5\cos 10g_0.$$

PERTURBATION EQUATIONS – long-term terms, distorted components

Solutions:

- Perturbed form of O-C:

$$\int \frac{a^{3/2}}{\mu^{1/2}} \frac{(1-e^2)^{3/2}}{[1+e\cos(u-\omega)]^2}\,\mathrm{d}u$$

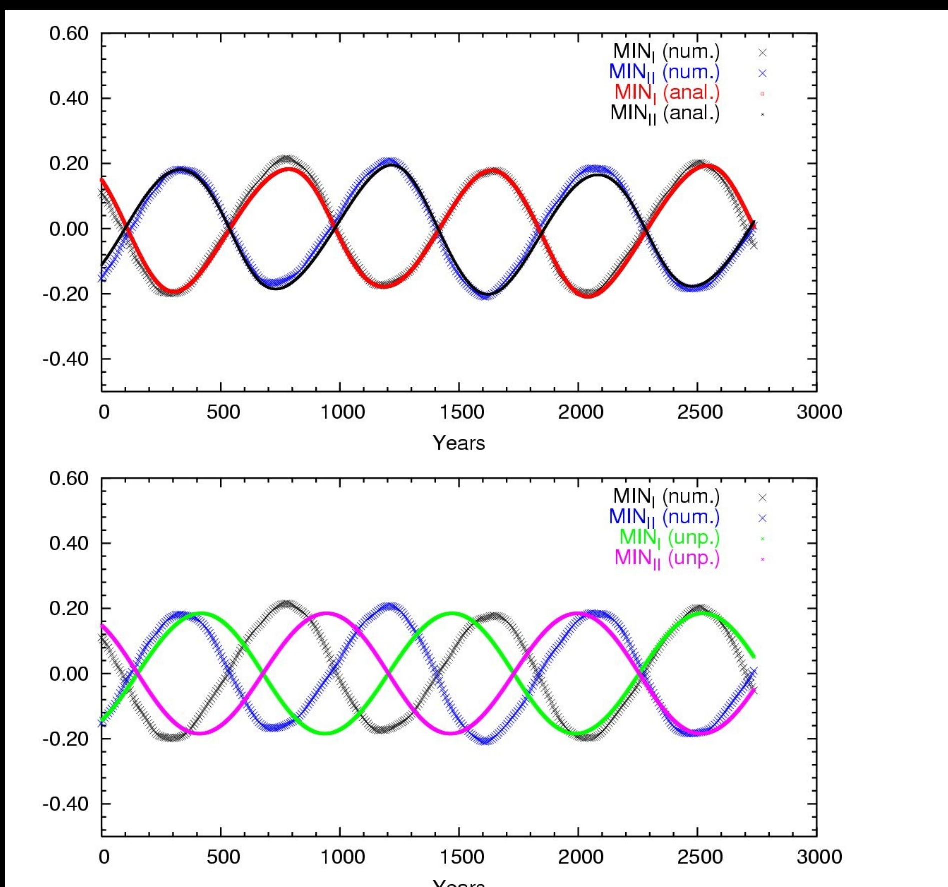
Includes long-term perturbations in all the orbital elements, as well as direct perturbations in mean motion

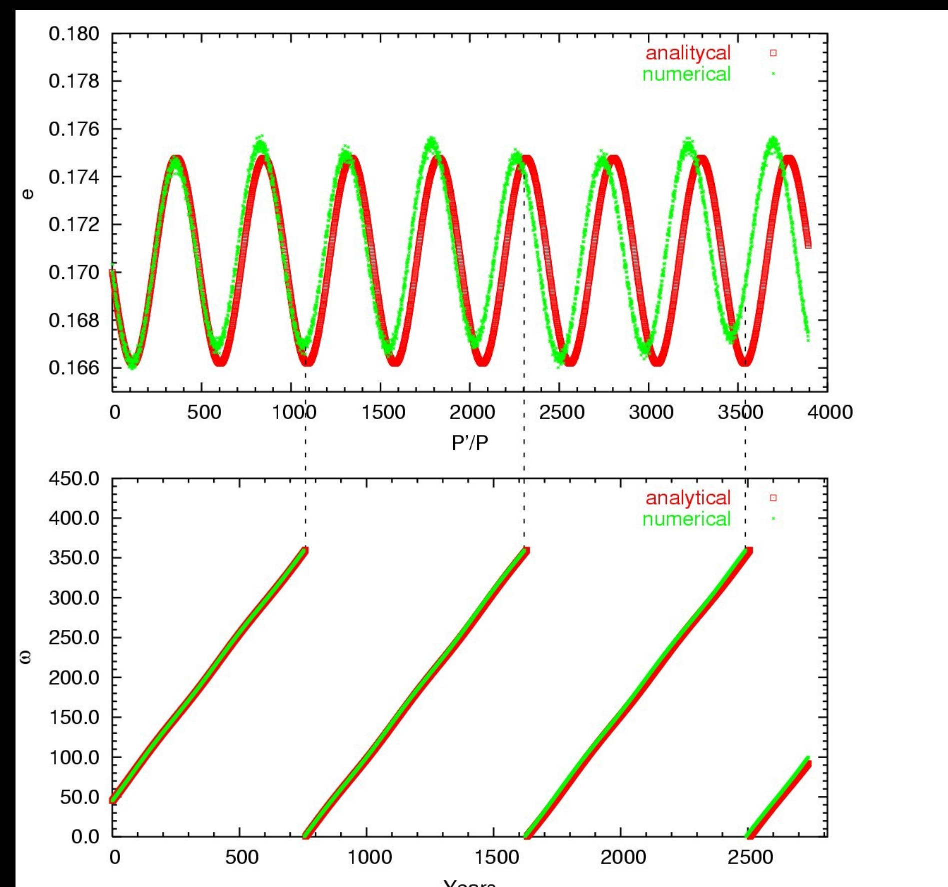
$$\int \frac{a^{3/2}}{\mu^{1/2}} \frac{(1-e^2)^{3/2}}{[1+e\cos(u-\omega)]^2}\,\frac{\rho_1^2}{c_1}\dot{\Omega}\cos i\,\mathrm{d}u$$

$$\begin{aligned}
\frac{2\pi}{P}O-C = {} & V_{100}\cos[\omega_0+(1+\mathcal{U})\Pi(u-u_0)] + V_{200}\sin[2\omega_0+(2+2\mathcal{U})\Pi(u-u_0)] + V_{300}\cos[3\omega_0+(3+3\mathcal{U})\Pi(u-u_0)] + \\
& +V_{101}\cos[\omega_0+h_0+(1+\mathcal{U}+\mathcal{H})\Pi(u-u_0)] + V_{10-1}\cos[\omega_0-h_0+(1+\mathcal{U}-\mathcal{H})\Pi(u-u_0)] + \\
& +V_{102}\cos[\omega_0+2h_0+(1+\mathcal{U}+2\mathcal{H})\Pi(u-u_0)] + V_{10-2}\cos[\omega_0-h_0+(1+\mathcal{U}-2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{103}\cos[\omega_0+3h_0+(1+\mathcal{U}+3\mathcal{H})\Pi(u-u_0)] + V_{10-3}\cos[\omega_0-3h_0+(1+\mathcal{U}-3\mathcal{H})\Pi(u-u_0)] + \\
& +V_{120}\cos[3\omega_0-u_{\mathrm{m}0}+(3+\mathcal{U})\Pi(u-u_0)] + V_{1-20}\cos[\omega_0-2u_{\mathrm{m}0}+(1-\mathcal{U})\Pi(u-u_0)] + \\
& +V_{121}\cos[3\omega_0-u_{\mathrm{m}0}+h_0+(3+\mathcal{U}+\mathcal{H})\Pi(u-u_0)] + V_{12-1}\cos[3\omega_0-u_{\mathrm{m}0}-h_0+(3+\mathcal{U}-\mathcal{H})\Pi(u-u_0)] + \\
& +V_{1-21}\cos[\omega_0-2u_{\mathrm{m}0}-h_0+(1-\mathcal{U}-\mathcal{H})\Pi(u-u_0)] + V_{1-2-1}\cos[\omega_0-2u_{\mathrm{m}0}+h_0+(1-\mathcal{U}+\mathcal{H})\Pi(u-u_0)] + \\
& +V_{122}\cos[3\omega_0-u_{\mathrm{m}0}+2h_0+(3+\mathcal{U}+2\mathcal{H})\Pi(u-u_0)] + V_{12-2}\cos[3\omega_0-u_{\mathrm{m}0}-2h_0+(3+\mathcal{U}-2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{1-22}\cos[\omega_0-2u_{\mathrm{m}0}-2h_0+(1-\mathcal{U}-2\mathcal{H})\Pi(u-u_0)] + V_{1-2-2}\cos[\omega_0-2u_{\mathrm{m}0}+2h_0+(1-\mathcal{U}+2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{123}\cos[3\omega_0-u_{\mathrm{m}0}+3h_0+(3+\mathcal{U}+3\mathcal{H})\Pi(u-u_0)] + V_{12-3}\cos[3\omega_0-u_{\mathrm{m}0}-2h_0+(3+\mathcal{U}-3\mathcal{H})\Pi(u-u_0)] + \\
& +V_{1-23}\cos[\omega_0-2u_{\mathrm{m}0}-3h_0+(1-\mathcal{U}-3\mathcal{H})\Pi(u-u_0)] + V_{1-2-3}\cos[\omega_0-2u_{\mathrm{m}0}+3h_0+(1-\mathcal{U}+3\mathcal{H})\Pi(u-u_0)] + \\
& +V_{140}\cos[5\omega_0-u_{\mathrm{m}0}+(5+\mathcal{U})\Pi(u-u_0)] + V_{1-40}\cos[3\omega_0-4u_{\mathrm{m}0}+(3-\mathcal{U})\Pi(u-u_0)] + \\
& +V_{201}\sin[2\omega_0+h_0+(2+2\mathcal{U}+\mathcal{H})\Pi(u-u_0)] + V_{20-1}\sin[2\omega_0-h_0+(2O_0+2\mathcal{U}-\mathcal{H})\Pi(u-u_0)] + \\
& +V_{202}\sin[2\omega_0+2h_0+(2+2\mathcal{U}+2\mathcal{H})\Pi(u-u_0)] + V_{20-1}\sin[2\omega_0-2h_0+(2+2\mathcal{U}-2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{203}\sin[2\omega_0+3h_0+(2+2\mathcal{U}+3\mathcal{H})\Pi(u-u_0)] + V_{20-1}\sin[2\omega_0-3h_0+(2+2\mathcal{U}-3\mathcal{H})\Pi(u-u_0)] + \\
& +V_{220}\sin[4\omega_0-2u_{\mathrm{m}0}+(4+2\mathcal{U})\Pi(u-u_0)] + V_{2-20}\sin[2u_{\mathrm{m}0}+2\mathcal{U}\Pi(u-u_0)] + \mathcal{O}(e,E)^4 + \\
& +V_{000}\Pi(u-u_0) + V_{001}\sin[h_0+\mathcal{H}\Pi(u-u_0)] + V_{002}\sin[2h_0+2\mathcal{H}\Pi(u-u_0)] + V_{003}\sin[3h_0+3\mathcal{H}\Pi(u-u_0)] + \\
& +V_{021}\sin[2\omega_0-2u_{\mathrm{m}0}+h_0+(2+\mathcal{H})\Pi(u-u_0)] + V_{0-21}\sin[\omega_0-u_{\mathrm{m}0}-h_0+(2-\mathcal{H})\Pi(u-u_0)] + \\
& +V_{022}\sin[2\omega_0-2u_{\mathrm{m}0}+2h_0+(2+2\mathcal{H})\Pi(u-u_0)] + V_{0-22}\sin[\omega_0-u_{\mathrm{m}0}-2h_0+(2-2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{023}\sin[2\omega_0-2u_{\mathrm{m}0}+3h_0+(2+3\mathcal{H})\Pi(u-u_0)] + V_{0-23}\sin[\omega_0-u_{\mathrm{m}0}-3h_0+(2-3\mathcal{H})\Pi(u-u_0)] + \\
& +V_{041}\sin[4\omega_0-4u_{\mathrm{m}0}+h_0+(4+\mathcal{H})\mathcal{G}] + V_{0-41}\sin[4\omega_0-4u_{\mathrm{m}0}-h_0+(4-\mathcal{H})\Pi(u-u_0)] + \\
& +V_{042}\sin[4\omega_0-4u_{\mathrm{m}0}+2h_0+(4+2\mathcal{H})\mathcal{G}] + V_{0-42}\sin[4\omega_0-4u_{\mathrm{m}0}-2h_0+(4-2\mathcal{H})\Pi(u-u_0)] + \\
& +V_{043}\sin[4\omega_0-4u_{\mathrm{m}0}+3h_0+(4+3\mathcal{H})\mathcal{G}] + V_{0-43}\sin[4\omega_0-4u_{\mathrm{m}0}-3h_0+(4-3\mathcal{H})\Pi(u-u_0)] + \ldots
\end{aligned} \qquad (745)$$

PERTURBATION EQUATIONS – long-term terms, distorted components

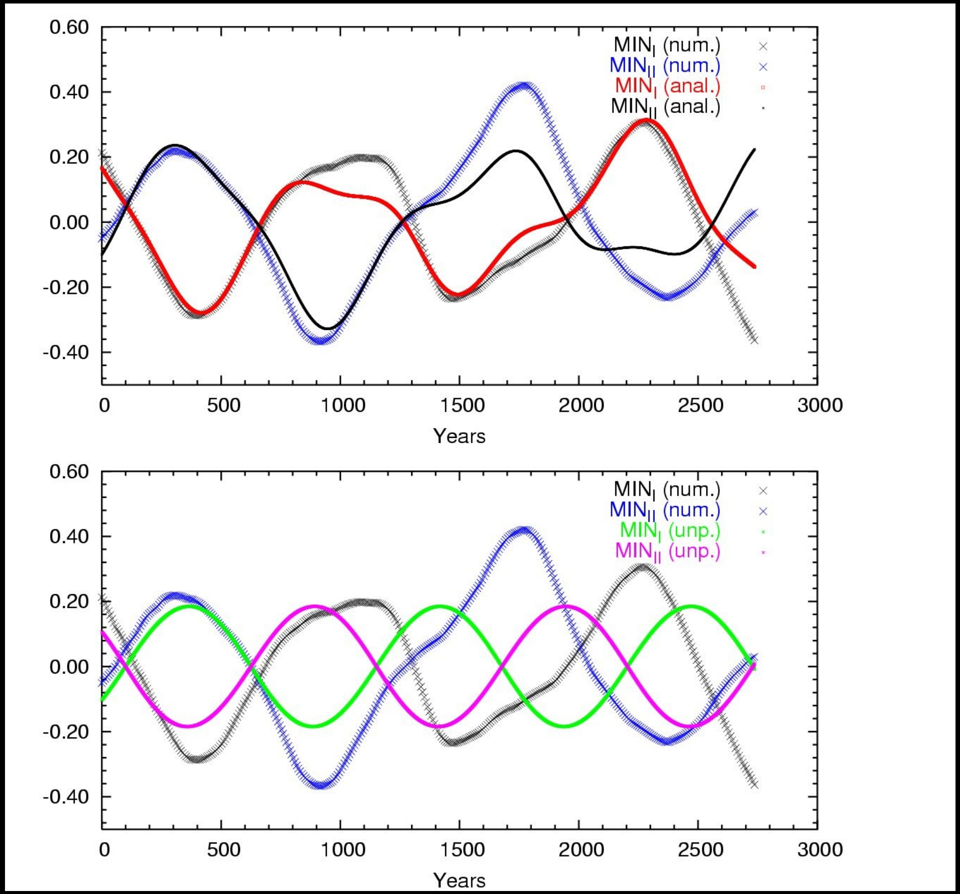
Mutual inclination: 20 deg

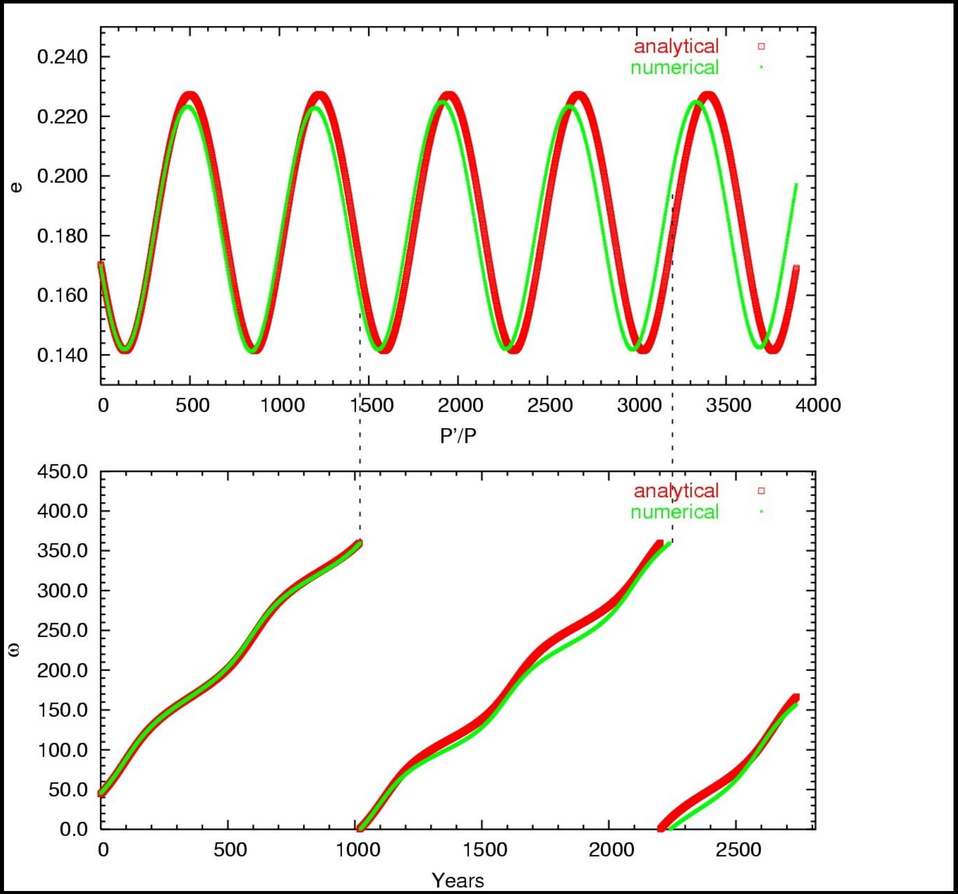




PERTURBATION EQUATIONS – long-term terms, distorted components

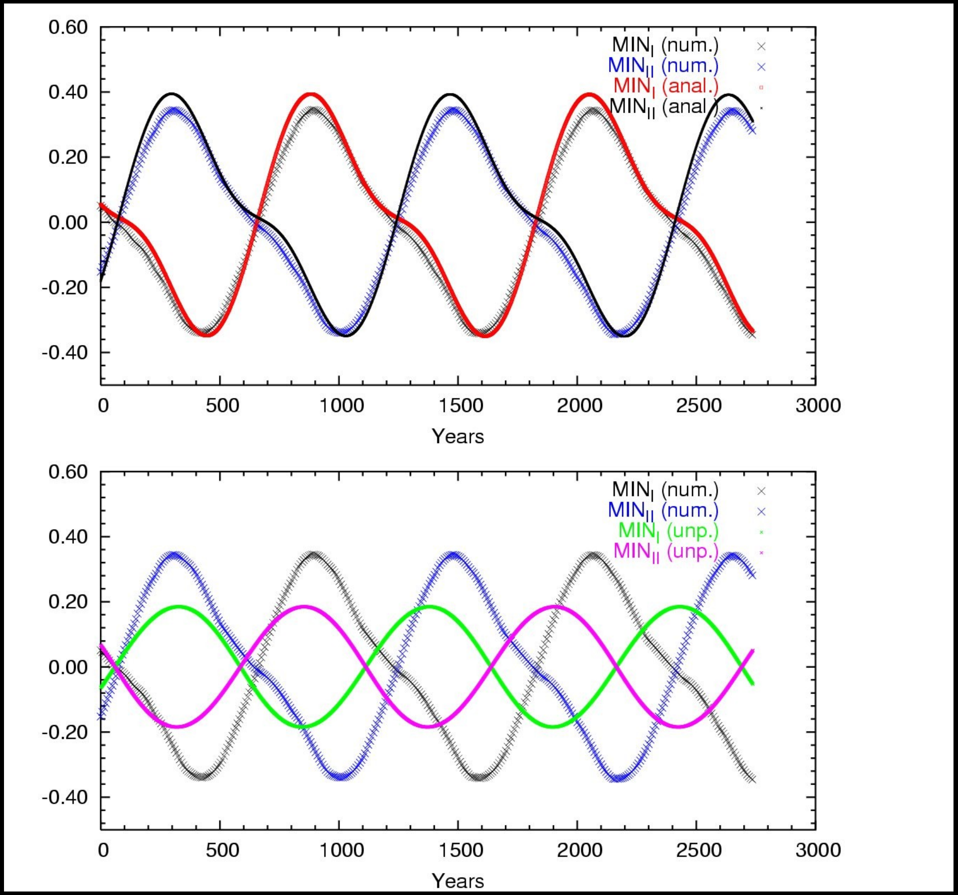
Mutual inclination: 60 deg

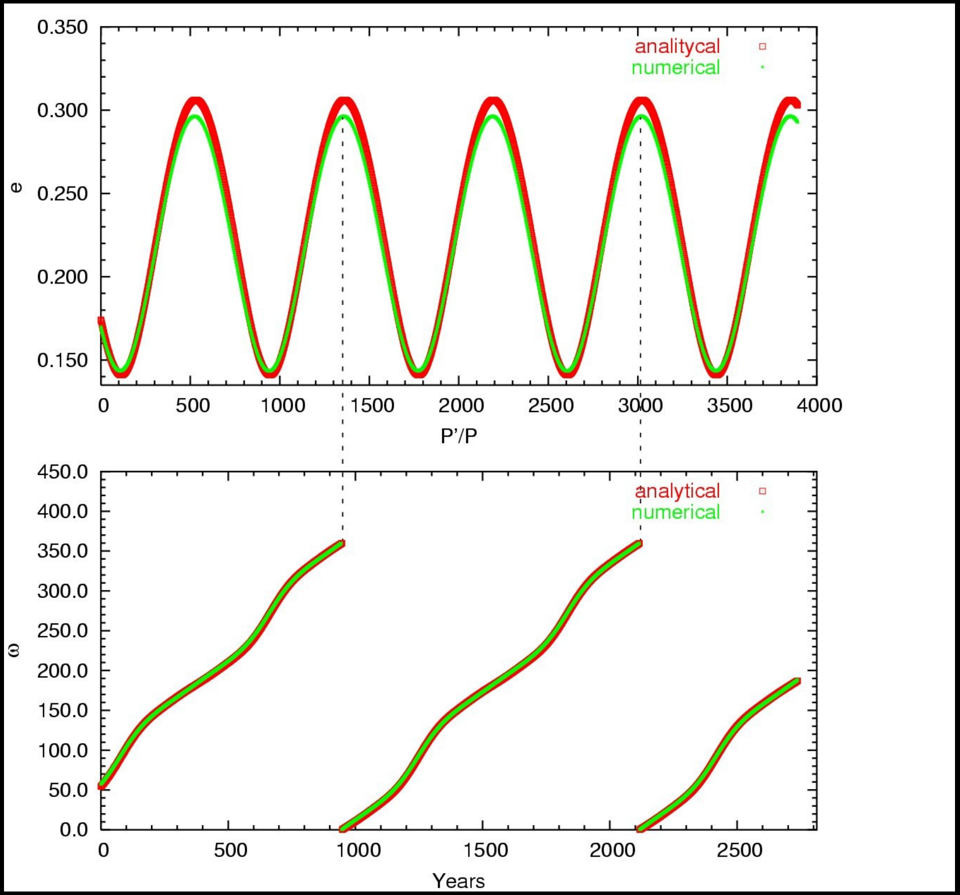




PERTURBATION EQUATIONS – long-term terms, distorted components

Mutual inclination: 89 deg





PERTURBATION EQUATIONS – long-term terms, distorted components

Analytical study on long-term perturbations in hierarchical triple systems with distorted components - Conclusions, future steps

Perturbations force significant variations in the apsidal motion period, as well as in eccentricity

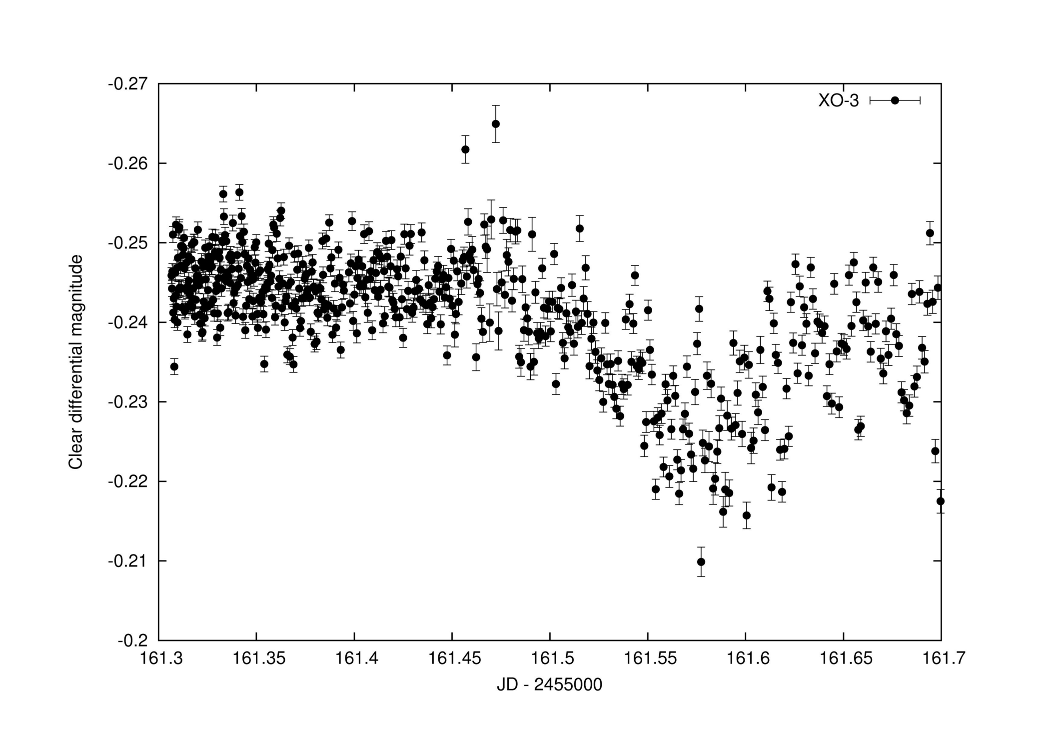
→ *new terms with large amplitude in the O-C curve, which should be considered at the determination of the speed of the apsidal motion (which sometimes determined simply from the slope of the observed short section of the O-C curve*

→ *variation of eccentricity can be determined from spectroscopy and/or accurate photometry. Combining these with the O-C the spatial orientation of the orbits might be calculated*

↓

Next step: to formulate the expressions for practical use (everything is ready for this).

THANK YOU FOR YOUR ATTENTION!



borko@alcyone.bajaobs.hu